

# A THERMODYNAMICAL FORMALISM FOR MONGE-AMPÈRE EQUATIONS, MOSER-TRUDINGER INEQUALITIES AND KÄHLER-EINSTEIN METRICS

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**ABSTRACT.** We develop a variational calculus for a certain free energy functional on the space of all probability measures on a Kähler manifold  $X$ . This functional can be seen as a generalization of Mabuchi's  $K$ -energy functional and its twisted versions to more singular situations. Applications to Monge-Ampère equations of mean field type, twisted Kähler-Einstein metrics and Moser-Trudinger type inequalities on Kähler manifolds are given. Tian's  $\alpha$ -invariant is generalized to singular measures, allowing in particular a proof of the existence of Kähler-Einstein metrics with positive Ricci curvature that are singular along a given anti-canonical divisor (as conjectured very recently by Donaldson). As another application we partially confirm a well-known conjecture in Kähler geometry by showing that if the Calabi flow in the (anti-) canonical class exists for all times then it converges to a Kähler-Einstein metric, when a unique one exists. Applications to the probabilistic/statistical mechanical approach to the construction of Kähler-Einstein metrics, very recently introduced by the author, will appear elsewhere.

## CONTENTS

1. Introduction	1
2. Functionals on the spaces of probability measures and $\omega$ -psh functions and Legendre duality	12
3. Monge-Ampère mean field equations and Moser-Trudinger type inequalities	21
4. The (twisted) Kähler-Einstein setting	33
5. Log Fano manifolds and Donaldson's equation	37
6. Appendix: $\alpha$ -invariants (log canonical thresholds) for pairs	44
References	46

## 1. INTRODUCTION

One of the motivations for the present paper comes from the probabilistic approach to Kähler-Einstein metrics very recently introduced in [8]. In op. cit. the relations to physics were emphasized (Euclidean gravity and fermion-boson correspondences) and a heuristic argument was given for the convergence of the statistical mechanics model in the thermodynamical limit. One of the aims of the present paper, which can be seen as the first part in a forthcoming series, is to develop the *variational calculus* needed for a rigorous investigation of the thermodynamical limit referred to above. However the main results to be proved also have an independent interest in Kähler-Einstein geometry (notably to the convergence of the Calabi flow and a conjecture of Donaldson concerning Kähler-Einstein metrics on

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*Key words and phrases.* Monge-Ampère equation, Kähler-Einstein manifolds, Variational methods (MSC 2010: 32Q20, 32W20, 35A15).

Fano manifolds which are singular along a divisor) and more generally in the context of complex Monge-Ampère equations and Moser-Trudinger type inequalities.

This work can also be seen as a development of the variational approach to complex Monge-Ampère equations recently introduced in [10]. The main role will be played by a certain functional  $F(\mu)$  on the space of all probability measures on a Kähler manifold that in the thermodynamical limit, referred to above, arises as the limiting *free energy functional*. We will be particularly interested in the *optimizers* of  $F(\mu)$  (as explained in [8] they determine the limiting equilibrium measures of the statistical mechanical model).

Using Legendre transforms the functional  $F(\mu)$  will be related to the another functional  $\mathcal{G}(u)$  on the space of all singular Kähler potentials (i.e.  $\omega$ -psh functions), which played a leading role in [10]. As it will turn out the free energy functional  $F(\mu)$  can, in the particular case when the Kähler class is proportional to the canonical class, be identified with *Mabuchi's K-energy functional*, which plays a leading role in Kähler-Einstein geometry. As for the functional  $\mathcal{G}(u)$  it generalizes a functional introduced by Ding [39] in Kähler-Einstein geometry that we following [73] will refer to as the *Ding functional*.

From the point of view of Kähler geometry the main conceptual contribution of the present paper is to introduce a thermodynamical formalism for Kähler-Einstein geometry, which in mathematical terms amounts to a systematic use of convexity and Legendre transform arguments and to - which is closely related - develop a variational calculus for the Mabuchi functional which demands a minimum of regularity assumptions, namely finite (pluricomplex) energy and finite entropy.

**1.1. The setup.** Let  $(X, \omega)$  be an  $n$ -dimensional compact complex manifold with Kähler form  $\omega$  and fix a probability measure  $\mu_0$  on  $X$  and non-zero real parameter  $\beta$  (which plays the role of the inverse temperature in the statistical mechanical setup in [8]). To the triple  $(\omega, \mu_0, \beta)$  we will associate a Monge-Ampère equation, as well as two functionals. Before continuing it should be emphasized that only the Kähler class  $[\omega] \in H^{1,1}(X)$  defined by the fixed Kähler form  $\omega$  will be relevant and one may as well fix any other smooth and, possibly non-positive, representative  $\omega' \in [\omega]$ . We let  $d^c := i(-\partial + \bar{\partial})/4\pi$ , so that  $dd^c = \frac{i}{2\pi}\partial\bar{\partial}$  and denote by  $V$  the volume of  $(X, \omega)$ , i.e.  $V = \int_X \omega^n/n!$  which by Stokes theorem is an invariant of the class  $[\omega]$ .

*The Monge-Ampère mean field equation.* This is the following equation

$$(1.1) \quad \frac{(\omega + dd^c u)^n}{Vn!} = \frac{e^{\beta u} \mu_0}{\int_X e^{\beta u} \mu_0}$$

for an  $\omega$ -psh function  $u$  on  $X$ , i.e.

$$(1.2) \quad \omega_u := \omega + dd^c u \geq 0$$

in the sense of currents. The integral in the equation has been inserted to ensure invariance under the additive action of  $\mathbb{R}$  (removing gives an equivalent equation) and hence the equation descends to the space of all positive currents in the class  $[\omega]$ .

The equation above generalizes the mean field equations extensively studied on a Riemann surface, i.e the case when  $n = 1$  (see the book [81] and references therein). It should be interpreted in the weak sense of pluripotential theory as recalled in section 2. More precisely, we will assume that the fixed measure  $\mu_0$  has finite energy and we will look for finite energy solutions. These energy notions will be recalled in section 2 - for the moment we just point out that in the case of a Riemann surface

they coincide with the classical notions of Dirichlet and logarithmic energy. But it should be kept in mind that the situation when  $n > 1$  is considerably more non-linear as the Monge-Ampère operator is fully non-linear, as opposed to the Laplace operator and moreover the non-linear constraint 1.2 has to be imposed a priori.

One of the main cases that we will be interested in is when  $\mu_0$  is a volume form and then we will simply look for *smooth* solutions of the equation 1.1 satisfying

$$\omega_u := \omega + dd^c u > 0,$$

which means that  $u$  is a *Kähler potential* for the Kähler metric  $\omega_u$  in the cohomology class  $[\omega]$ . An interesting case when  $\mu_0$  is singular is obtained by letting  $\mu_0$  be supported on a real hypersurface  $M \subset X$  (compare prop 3.1).

Interestingly, the equation 1.1 also has a natural interpretations for  $\beta = 0$ , as well as  $\beta = \infty$ . Indeed, for  $\beta = 0$  this is clearly the *inhomogeneous Monge-Ampère equation* and for  $\beta = \infty$  it may be interpreted as a *free boundary value problem* for the Monge-Ampère equation (see Theorem 3.12).

*The Kähler-Einstein setting.* The case of main interest in Kähler geometry arises when the class  $[\omega]$  is a non-zero multiple of the *canonical class*, i.e. the first Chern class of the canonical line bundle  $K_X := \Lambda^n(TX^*)$ :

$$[\omega] = \beta c_1(K_X)$$

(after scaling we may and will assume that  $\beta = \pm 1$ ) and when the fixed Kähler form  $\omega$  and measure  $\mu_0$  are related by

$$\mu_0 = e^{-h_\omega} \omega^n / V$$

for the Ricci potential  $h_\omega$  of the fixed Kähler metric  $\omega$ . Then the equation 1.1 is equivalent to the *Kähler-Einstein equation*

$$\text{Ric}\omega = -\beta\omega$$

where  $\text{Ric}\omega$  denotes the Ricci form defined by the Ricci curvature of the Riemannian metric determined by  $\omega$ . By the seminal theorem of Aubin [2] and Yau [93] such a Kähler-Einstein metric always exists in the case when  $\beta \geq 0$ . But it is well-known that there are obstructions to the existence of Kähler-Einstein metrics in the case when  $\beta < 0$ , i.e. when  $X$  is a Fano manifold. An influential conjecture introduced by Yau and later extended by Tian, Donaldson and others (see [43, 86] and references therein) formulates these obstructions in terms of an algebro-geometric notion of *stability* (in the sense of Geometric Invariant Theory). Even though there has been tremendous progress on this conjecture, which was settled on complex surfaces by Tian [84], it is still open in dimension  $n \geq 3$ . However, as shown by Tian (see [86]) there is a stronger *analytic* notion of stability which is equivalent to the existence of a Kähler-Einstein metric in the class  $\beta c_1(K_X)$ , namely the *properness* of Mabuchi's *K-energy functional*  $\mathcal{K}$  (which in this case turns out to be equivalent to the *coercivity* of the functional [70]). The functional  $\mathcal{K}$  is defined on the subspace of all Kähler metrics in  $\beta c_1(K_X)$  and its critical points are precisely the Kähler-Einstein metrics. In the case of a general class  $[\omega]$  and volume form  $\mu_0$  the equation 1.1 is equivalent to a *twisted* Kähler-Einstein equation (see section 4).

*The functionals  $F_\beta$  and  $\mathcal{G}_\beta$ .* We will introduce a functional  $F_\beta(\mu)$  to be referred to as the *free energy functional* on the space  $E_1(X, \omega)$  of all probability measures on  $X$  with finite energy. In the Kähler-Einstein setting this functional (on the subspace of volume forms) will be identified with (minus) Mabuchi's *K-energy*  $\mathcal{K}$  functional

referred to above. In general the free energy functional  $F$  may be defined by the sum

$$(1.3) \quad F_\beta = E_\omega + \frac{1}{\beta} D_{\mu_0}$$

where  $E_\omega(\mu)$  is the (pluricomplex) *energy* of the probability measure  $\mu$  introduced in [10] and  $D_{\mu_0}$  is its *entropy* relative to  $\mu$ . In the smooth case the energy  $E_\omega(\mu)$  is explicitly defined as follows:

$$E_\omega(\mu) = \mathcal{E}_\omega(u_\mu) - \int_X u_\mu \mu$$

(which is non-negative if  $\omega \geq 0$ ), where  $\mathcal{E}_\omega(u)$  is the following functional well-known in Kähler geometry:

$$\mathcal{E}_\omega(u) := \frac{1}{(n+1)!V} \sum_{j=0}^n \int_X \omega_u^j \wedge (\omega)^{n-j}$$

and  $u_\mu$  is a potential of  $\mu$ , i.e a. Kähler potential such that

$$\frac{(\omega + dd^c u_\mu)^n}{Vn!} = \mu$$

In the general singular setting we will work with the *finite energy spaces*  $E_1(X, \omega)$  and  $\mathcal{E}^1(X, \omega)$ , defined as the subspaces where the functionals  $E_\omega(\mu)$  and  $\mathcal{E}_\omega(u)$  are, respectively, finite.

The well-known *relative entropy functional*  $D_{\mu_0}$  (which is *minus* the physical entropy) is defined by

$$D_{\mu_0}(\mu) := \int_X \log\left(\frac{\mu}{\mu_0}\right) \mu \geq 0$$

if  $\mu$  is absolutely continuous with respect to  $\mu_0$  and otherwise equal to infinity. Formally, the critical points of  $F_\beta$  are precisely the measures  $\mu$  whose potential  $u_\mu$  satisfies the equation 1.1.

It will also be important to relate the free energy functional  $F_\beta(\mu)$  to another functional  $\mathcal{G}_\beta(u)$  defined on the space  $\mathcal{E}^1(X, \omega)$  :

$$\mathcal{G}_\beta(u) := \mathcal{E}_\omega(u) - \frac{1}{\beta} \log \int_X e^{\beta u} \mu_0$$

In the Kähler-Einstein setting the functional  $\mathcal{G}(u)$  coincides (up to a sign) with the *Ding functional* [39, 40] (in the book [86],  $\mathcal{G}_{-1}(u) = -F_\omega(\omega_u)$ ). Formally, its critical points also coincide with solutions to the equation 1.1. As will be explained in section 2 the functional  $\mathcal{G}_\beta$  may be obtained from the functional  $F_\beta$  by replacing  $E$  and  $\frac{1}{\beta}D$  by their Legendre transforms:

$$\mathcal{G}_\beta = E^* + \left(\frac{1}{\beta}D\right)^*$$

on  $\mathcal{E}^1(X, \omega)$  when  $\beta > 0$  and a similar expression holds when  $\beta < 0$ , but with a crucial sign difference.

The functionals  $F_\beta$  and  $\mathcal{G}_\beta$  have an independent analytical interest when  $\beta < 0$ . For example, on a Riemann surface their boundedness from above is equivalent to a *logarithmic Hardy-Sobolev inequality* and *Moser-Trudinger inequality*, respectively (which in turn imply various limiting Sobolev inequalities) [6, 25].

## 1.2. Statement of the main results.

*Monge-Ampère mean field equations and Moser-Trudinger type inequalities.* We will start by relating properties of the free energy functional  $F_\beta$  to properties of the functional  $\mathcal{G}_\beta$ . In particular, in the analytically most challenging case when  $\beta < 0$  the main properties that will be obtained are summarized in the following Theorem (see section 2.7 for the definition of properness and coercivity in this context).

**Theorem 1.1.** *For any given measure  $\mu_0$  of finite energy and number  $\beta < 0$  we have*

$$(1.4) \quad \sup_{\mu \in E_1(X)} F_\beta = \sup_{u \in \mathcal{E}^1(X, \omega)} \mathcal{G}_\beta$$

and

$$(1.5) \quad F_\beta\left(\frac{(\omega + dd^c u)^n}{Vn!}\right) \leq \mathcal{G}_\beta(u)$$

for any  $u \in \mathcal{E}^1(X, \omega)$  with equality iff  $u$  is a solution to the Monge-Ampère mean field equation 1.1. Moreover, the functional  $F_\beta$  is coercive iff  $\mathcal{G}_\beta$  is.

Before continuing we point out that in the Kähler-Einstein setting and when  $u$  is assumed to be a Kähler potential - so that  $\beta F_\beta$  may be identified with Mabuchi's K-energy functional and  $\beta \mathcal{G}_\beta(u)$  is the Ding functional - the content of the previous theorem was previously known. Indeed, the equality 1.4 was established by Li [62], who used the Kähler-Ricci flow and Perelman's deep estimates and by Rubinstein [71, 73], using the Ricci iteration. As for the inequality 1.5 it follows from identities of Bando-Mabuchi [3], while the coercivity statement only has a rather involved and indirect proof (see section 3.2.2 for further discussion and references). The present proof uses a simple Legendre duality argument and has the virtue of being valid in the general singular setting.

Combining the properties 1.4 and 1.5 above with the variational approach introduced in [14] is the key to the proof of the following general existence and convergence result.

**Theorem 1.2.** *Let  $(X, \omega)$  be a compact Kähler manifold and let  $\mu_0$  be a probability measure on  $X$  of finite energy.*

- When  $\beta > 0$  the free energy functional  $F_\beta(\mu)$  admits a unique minimizer  $\mu$  on the space  $E_1(X, \omega)$  of finite energy probability measures on  $X$ . Its potential  $u_\mu$  is the unique solution (mod  $\mathbb{R}$ ) of the Monge-Ampère mean field equation 1.1.
- When  $\beta < 0$  and the free energy functional  $F_\beta(\mu)$  is assumed bounded from above on the space  $E_1(X, \omega)$  any maximizer  $\mu$  (if it exists) has a potential  $u_\mu$  solving the Monge-Ampère mean field equation 1.1. Moreover, under the stronger assumption that  $F_{\beta-\delta}$  be bounded from above for some  $\delta > 0$  (or equivalently, if  $-F_\beta$  is coercive with respect to energy) a maximizer does exist.

More generally, if the free energy functional  $F_\beta$  is coercive on  $E_1(X, \omega)$  with respect to energy, then any sequence  $\mu_j$  in  $E_1(X, \omega)$  such that  $F(\mu_j)$  converges to the optimal value of  $F$  converges (perhaps after passing to a subsequence if  $\beta < 0$ ) to an optimizer  $\mu$  of  $F$ . In the case when  $\mu_0 = f dV$  for a volume form  $dV$  on  $X$  and  $f \in L^p(X, dV)$  for some  $p > 1$  the assumptions about coercivity above may be replaced by properness.

Before continuing we make some remarks about the coercivity vs. properness assumptions in the previous theorem. First, in the case when  $\beta > 0$  the free

functional  $F_\beta$  is trivially coercive. Secondly, when  $\beta < 0$  the coercivity assumptions may be replaced by properness as long as  $\mu_0$  satisfies a qualitative Moser-Trudinger type inequality (see 3.14).

In the case when  $\mu_0$  is a volume form the weak solutions of the equation 1.1 produced above are in fact *smooth* as follows from the work of Kolodziej [58] (which gives continuity of the solution) combined with the very recent result of Székelyhidi-Tosatti [79] which then gives higher order regularity, by smoothing with the associated parabolic flow (coinciding with the Kähler-Ricci flow in the Kähler-Einstein setting).

As pointed out above, in the Kähler-Einstein setting the existence result in the previous theorem was shown by Aubin and Yau in the case when  $\beta > 0$  and by Tian in the case when  $\beta < 0$ . The usual existence proofs are based on the continuity method which in the case when  $\beta = -1$  gives a path  $\omega_t$  such that

$$(1.6) \quad \text{Ric}\omega_t = t\omega_t + (1-t)\omega$$

deforming a given Kähler metric  $\omega_0$  with positive Ricci curvature  $\omega$  to a Kähler-Einstein metric.

However, in the general situation when  $\beta < 0$  it does not seem possible (even when  $\mu_0$  is a volume form) to use a continuity method as there is no general uniqueness result for the solutions (even modulo biholomorphisms), nor for the solutions of the linearized equations and hence the crucial openness property in the continuity method is missing in general.

The existence criterion in the next theorem is a generalization of a result of Tian [83], which concerned the Kähler-Einstein setting. It is formulated in terms of an invariant of a pair  $([\omega], \mu_0)$  which is a generalization of Tian's  $\alpha$ -invariant

$$\alpha([\omega], \mu_0) := \sup \left\{ \alpha : \exists C_\alpha : \int_X e^{-\alpha(u - \sup_X u)} \mu_0 \leq C_\alpha, \forall u \in PSH(X, \omega) \right\}$$

When  $\mu_0$  is any given volume form on  $X$  and the Kähler class  $[\omega] = c_1(L)$  is the first Chern class of an ample line bundle the corresponding invariant of the class  $[\omega]$  coincides with the algebro-geometrically defined *log canonical threshold of  $L$*  (which is precisely Tian's  $\alpha$ -variant when  $c_1(L) = -c_1(K_X)$ ). The case of a singular measure  $\mu_0$  was recently studied by Dinh-Nguyễn-Sibony in complex dynamics [38]. In their terminology,  $\alpha([\omega], \mu_0) > 0$  precisely when the measure  $\mu_0$  is of *global moderate growth* (with respect to the Kähler class  $[\omega]$ ). As shown in [38] this condition in particular holds when the potential of  $\mu_0 = \omega_{u_0}^n/n!$  is Hölder continuous and in particular for many of the equilibrium measures which arise as limits in complex dynamics and whose supports typically are fractal sets.

**Theorem 1.3.** *Let  $(X, \omega)$  be a compact Kähler manifold and let  $\mu_0$  be a probability measure on  $X$  of finite energy. If the parameter  $\beta := -\gamma$  (with  $\gamma > 0$ ) satisfies the bound*

$$(1.7) \quad \gamma < \alpha \frac{n+1}{n}$$

where  $\alpha$  is the  $\alpha$ -invariant of the pair  $([\omega], \mu_0)$ , then the following holds:

- Both the functionals  $F_\beta$  and  $\mathcal{G}_\beta$  are bounded, i.e. the corresponding logarithmic Hardy-Sobolev and Moser-Trudinger type inequalities hold
- There is a maximizer  $\mu$  of  $F_\beta$ . Moreover, its potential  $u_\mu$  maximizes  $\mathcal{G}_\beta$  and solves the Monge-Ampère mean field equation 1.1.

The point is that when  $\beta$  satisfies the open condition 1.7  $F_\beta$  is in fact coercive. Next, specializing Theorem 1.3 to a Riemann surfaces with  $\mu_0$  a Frostman measure gives the following

**Corollary 1.4.** *Let  $X$  be a Riemann surface and  $\mu_0$  a probability measure such that*

$$\mu_0(B_r) \leq Cr^d$$

*for some positive constants  $C$  and  $d$ , for any local coordinate ball  $B_r$  of sufficiently small radius  $r$ . Then, for any  $\delta > 0$  there is a constant  $C_\delta$  such that*

$$\log \int_X e^u \mu_0 \leq \frac{(d+\delta)}{2} \frac{1}{4} \int_X du \wedge d^c u + C_\delta$$

*for any smooth function  $u$  on  $X$  normalized so that  $\int_X u \omega = 0$  for a fixed measure  $\omega$  on  $X$ .*

It seems likely that one can take  $\delta = 0$  in the previous corollary by further studying the blow-up behavior of the functional  $\mathcal{G}_{\alpha-\delta}$  when  $\delta \rightarrow 0$ . Indeed, when  $\mu$  is a volume (are rather area) form setting  $\delta = 0$  does give an optimal inequality according to Fontana's generalization [48] of Moser's inequality on the two-sphere  $S^2$ . Even though formulated for Riemann surfaces without boundary the corollary above also contains the analogous statement on any compact Riemann surface  $Y$  with smooth boundary  $\partial Y$  if one demands, as usual, that  $u = 0$  on  $\partial Y$  (see remark 3.11). In particular, taking  $Y$  as a domain in  $\mathbb{R}^2$  one gets a weaker version of a recent result of Cianchi [32] who proved the corresponding inequality with  $\delta = 0$ , using completely different methods. This latter result has very recently been further developed, still in the setting of  $\mathbb{R}^2$ , by Morpurgo-Fontana [49], building on Adam's seminal work [1].<sup>1</sup>

*The (twisted) Kähler-Einstein setting and log Fano manifolds.* In the Kähler-Einstein setting the functional

$$(1.8) \quad \mathcal{K}(u) := \beta F_\beta \left( \frac{\omega_u^n}{Vn!} \right)$$

on the space  $\mathcal{H}(X, \omega)$  of Kähler potentials for  $[\omega] = \beta c_1(K_X)$  will be shown to coincide with *Mabuchi's  $K$ -energy functional*. This latter functional was first defined by Mabuchi [64] by the implicit property that its gradient at the point  $u$  in the infinite dimensional Riemannian manifold  $(\mathcal{H}(X, \omega), g_{sym})$  equipped with its negatively curved symmetric space metric (the Mabuchi-Semmes-Donaldson metric) [65], is the scalar curvature of the Kähler metric  $\omega_u$  minus its average. The explicit formula for  $\mathcal{K}$  obtained from 1.8 and the decomposition 1.3 coincides with a formula for  $\mathcal{K}$  due to Tian [87] (5.12). Tian's formula was generalized by Chen [26] who used it to define and study  $\mathcal{K}$  on potentials  $u$  such that  $\omega_u$  is locally bounded. More generally, the formula 1.8 allows one to extend the definition of  $\mathcal{K}$  to the space  $\mathcal{E}^1(X, \omega)$  of finite energy potentials and to deduce the following Corollary from Theorem 1.2:

**Corollary 1.5.** *Let  $u$  be an  $\omega$ -psh function with finite energy, i.e.  $u \in \mathcal{E}^1(X, \omega)$ . Then  $u$  minimizes (with a finite minimum) the generalized Mabuchi functional  $\mathcal{K}$  iff  $\omega_u$  is a Kähler-Einstein metric (and in particular smooth and non-degenerate).*

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<sup>1</sup>It was pointed out in [49] that the methods in [49] can be generalized to the setting of compact manifolds using pseudo-differential calculus - presumably such a generalization would lead to the sharp version of Cor 1.4 discussed above. Moreover, the results in [49] also give higher dimensional Moser-Trudinger type inequalities, but for other operators than the Monge-Ampère operator.

More over, if  $\mathcal{K}$  is assumed proper (wrt energy) and  $u_j$  is a sequence of finite energy potentials such that

$$\mathcal{K}(u_j) \rightarrow \inf_{\mathcal{H}(X, \omega)} \mathcal{K}$$

Then the currents  $\omega_{u_j}$  converge, after perhaps passing to a subsequence, to a Kähler-Einstein metric weakly.

This corollary generalizes a recent result of Chen-Tian-Zhou [30], saying that any maximizer  $u$  such that  $\omega_u$  has locally bounded coefficients is necessarily smooth and Kähler-Einstein. This latter result established a general conjecture of Chen in the case of a Kähler class  $[\omega]$  proportional to the canonical class. The proof in [30] is obtained by evolving the weak Kähler-Ricci flow developed in [30] with the maximizer  $u$  as initial data (see also [76, 79] for further developments of this flow). It should be pointed out that the minimal assumption of finite energy of the maximizer  $u$  in the assumptions in Corollary 1.5 is crucial as there seems to be no known way of controlling the a priori regularity of a general maximizer. In particular, this will allow us to apply the previous corollary to the Calabi flow.

The *Calabi flow* [20] is the following flow of Kähler metrics:

$$\frac{\partial \omega_t}{\partial t} = dd^c R_{\omega_t}$$

where  $R_{\omega_t}$  is the scalar curvature of the Kähler metric  $\omega_t$ , which is a highly non-linear 4th order parabolic PDE. It has been conjectured by Chen that the flow exists for all times and that it converges to a constant scalar curvature metric in  $[\omega]$  when such a metric exists (see for example [43]). In this direction we will prove the following

**Theorem 1.6.** *Let  $[\omega]$  be a Kähler class such that  $[\omega] = \beta c_1(K_X)$  for  $\beta \neq 0$ . In case  $\beta < 0$  we assume that  $X$  admits a Kähler-Einstein metric  $\omega_{KE}$  (and that  $H^0(TX) = \{0\}$ , so that  $\omega_{KE}$  is unique [3]). If the Calabi flow  $\omega_t$  exists for all times  $t \geq 0$ , then it converges weakly to the Kähler-Einstein metric, i.e.*

$$\omega_t \rightarrow \omega_{KE},$$

as  $t \rightarrow \infty$  holds in the weak topology of currents.

The existence and convergence of the Calabi flow on a Riemann surface was shown by Chrusciel [31]. In the general higher dimensional case almost all results are conditional. It was proved by Chen-He [29] that the Calabi flow exists as long as the Ricci curvature stays uniformly bounded. Moreover, they obtained the convergence towards an extremal metric (which in the case  $[\omega] = \beta c_1(K_X)$  is the Kähler-Einstein metric) under the extra assumption that the either the potential  $u_t$  be uniformly bounded along the flow (giving a convergence in the  $\mathcal{C}^\infty$ -topology) or the initial metric be sufficiently close to the Kähler-Einstein metric. See also [82] for results related to [29]. In [78] the case when  $X$  is a ruled surface was studied and the long-time existence of the Calabi flow (in a general Kähler class) and its convergence towards a minimizer of the Calabi functional was established for initial metrics satisfying a certain symmetry assumption.

The previous theorem should be viewed in the light of the corresponding result for the Kähler-Ricci flow in  $\beta c_1(K_X)$ . As shown by Cao [23] this latter flow exists for all times, regardless of the sign of  $\beta$ , and converges to the Kähler-Einstein metric when  $\beta < 0$ . However, the convergence towards a Kähler-Einstein metric (when it exists) was only proved recently by Tian-Zhou [88] using the deep estimates of Perelman.



The starting point of the proof of the previous theorem is the well-known fact that the Calabi flow on  $\mathcal{H}(X, \omega)$  is the downward gradient flow for the Mabuchi functional  $\mathcal{K}$  on the infinite dimensional Riemannian manifold  $(\mathcal{H}(X, \omega), g)$  where  $g$  is the Mabuchi-Semmes-Donaldson metric on the space of Kähler metrics in  $[\omega]$  [65]. As shown by Mabuchi [65] the functional  $\mathcal{K}$  is geodesically convex and some crucial strengthenings of this convexity due to Calabi-Chen [21] and Chen [28] will allow us to deduce the previous theorem from Corollary 1.5. In fact, the previous theorems extends to the setting of twisted Kähler-Einstein metrics as long as the twisting form  $\theta$  (see below) is non-negative (see Remark 4.5).

Next, specializing Theorem 1.3 to the case when  $\mu_0$  is a volume form gives the following

**Corollary 1.7.** *Let  $\gamma$  be a positive number and  $\theta$  a closed  $(1,1)$ -form on the  $n$ -dimensional compact complex manifold  $X$  such that the class  $-(\gamma c_1(K_X) + [\theta])$  in  $H^2(X, \mathbb{R})$  is Kähler (i.e. contains some Kähler form)*

- *If the  $\alpha$ -invariant of the class  $-(\gamma c_1(K_X) + [\theta])$  satisfies*

$$\alpha > \gamma \frac{n}{n+1}$$

*then the class contains a Kähler form  $\omega$  which solves the twisted Kähler-Einstein equation*

$$(1.9) \quad \text{Ric}\omega = \gamma\omega + \theta$$

*and which minimizes the twisted Mabuchi  $K$ -energy  $\mathcal{K}_\theta$ .*

- *More precisely, if  $(u_j)$  is a normalized asymptotically minimizing sequence for  $\mathcal{K}_\theta$  then any given  $L^1$ -accumulation point  $u_\infty$  of  $(u_j)$  is either the potential of a  $\theta$ -twisted Kähler-Einstein metric or  $u_\infty$  defines a Nadel type multiplier ideal sheaf, i.e.  $\int_X e^{-t\gamma u_\infty} dV = \infty$  for any  $t > \frac{n}{n+1}$ .*

The parameter  $\gamma$  may, of course, be set to one after scaling  $\omega$  but it has been included for later convenience. In the standard un-twisted case, i.e. when  $\theta = 0$  the first point in the previous corollary is due to Tian [83], who used the continuity method, which as explained above is not applicable in the general twisted setting. As for the second point above it generalizes a result of Nadel [68] and Demailly-Kollar [35] concerning the case when  $u_j$  is a subsequence of the curve  $u_t$  appearing in the continuity method (see remark 4.6) and hence the result in the second point above is new even when  $\theta = 0$ .

Twisted Kähler-Einstein metrics and the corresponding twisted Mabuchi  $K$ -energy recently appeared in the works of Fine [47] and Song-Tian [75] in their study of fibrations  $Y \rightarrow X$  (see also [77] for relations to stability). It should be pointed out that when  $\theta \geq 0$  the usual continuity method does apply, using that  $\text{Ric}\omega_\gamma > 0$  along the path  $\gamma \mapsto \omega_\gamma$  furnished by the corresponding continuity method:

$$\text{Ric}\omega_\gamma = \gamma\omega_\gamma + (1 - \gamma)\theta$$

Moreover, any twisted Kähler-Einstein metrics minimizes the twisted  $K$ -energy [77] (see Prop 4.3). However, for a twisting form  $\theta$  which is not semi-positive the minimizing property of the solution furnished by the corollary above is not automatic and moreover there are no uniqueness properties of the solutions (see the discussion and references on p. 65 in [81] for the Riemann surface case).

*Log Fano manifolds and Donaldson's equation.* In section 5 we will allow the twisting form  $\theta$  above to be singular, for example proportional to the current of integration along a divisor  $\Delta$  on  $X$ . The corresponding twisted setting then gives a way to associate Kähler-Einstein metrics to log pairs  $(X, \Delta)$ , where  $X$  is smooth and  $\Delta$  is a *sub klt* divisor (in the usual terminology of the minimal model program in algebraic geometry) and we will focus on the *log Fano* case, i.e.  $-c_1(K_X + \Delta) > 0$  (the case when  $c_1(K_X + \Delta) > 0$  and more generally when  $c_1(K_X + \Delta)$  is big and  $X$  may be singular was studied recently in [46] and [92] and the twisted singular case when  $\gamma = 0$  was already studied in Yau's seminal paper [93]). In this setting the corresponding alpha-invariant may be identified with the *log canonical threshold* of the pair  $(X, \Delta)$ . This leads to generalizations of results of Demailly-Kollar [35] concerning the Fano orbifold case (see Cor 5.2).

In the setting of log Fano manifolds we obtain in particular the following

**Theorem 1.8.** *Let  $X$  be a Fano manifold with a smooth irreducible anti-canonical divisor  $D$ . Let  $\gamma$  be a fixed parameter such that*

$$0 < \gamma < \Gamma := \frac{n+1}{n} \min \{ \alpha(-K_X), \alpha((-K_X)|_D) \},$$

(where  $\Gamma > 0$ ).

- *There is a smooth Kähler-Einstein metric  $\omega_\gamma$  on  $X - D$  such that  $\omega_\gamma$  has Hölder continuous local potentials on all of  $X$  and the following equation of currents holds globally on  $X$  (Donaldson's equation)*

$$(1.10) \quad \text{Ric}\omega_\gamma = \gamma\omega_\gamma + (1 - \gamma)\delta_D$$

where  $\delta_D$  denotes the current of integration along  $D$ . Moreover,  $\omega_\gamma \geq \omega$  for some Kähler form  $\omega$  on  $X$ .

- *the metric  $\omega_\gamma$  is unique and  $\gamma \mapsto \omega_\gamma$  ( $\gamma \in ]0, \Gamma[$ ) is a continuous curve in the space of Kähler currents on  $X$  and the restriction to  $X - D$  gives a continuous curve in the space of all Kähler forms on  $X - D$  equipped with the  $C^\infty$ -topology on compacts.*

The key point of the proof is to study how the  $\alpha$ -invariant (log canonical threshold) of the pair  $(X, (1-\gamma)D)$  depends on the parameter  $\gamma$ . This will allow us to show that the twisted Mabuchi  $K$ -energy  $\mathcal{K}_{(1-\gamma)D}$  is proper (and even coercive) when  $\gamma < \Gamma$ . Then the previous variational approach can be used to produce a weak solution to equation 1.10. From the properness we also deduce a global  $C^0$ -estimate which is used to prove the general regularity properties stated in the first two points of the theorem. As for the uniqueness it is deduced from Berndtsson's very recent generalized Bando-Mabuchi theorem [13], saying that uniqueness holds for solutions to equations of the form 1.10, given a smooth divisor  $D$ , unless there are non-trivial holomorphic vector fields on  $X$  tangent to  $D$ . In our case the non-existence of such vector fields follows from the properness of  $\mathcal{K}_{(1-\gamma)D}$ , which, as explained above, holds for any  $\gamma$  sufficiently small.

Theorem 1.8 was inspired by a very recent program of Donaldson for producing Kähler-Einstein metrics (under suitable stability assumptions) by first obtaining solutions to equation 1.10 for some  $\gamma = \gamma_0$  and then deforming  $\gamma$ . More precisely, in Step 1 in the notes [44], p.33, it was conjectured that there is a solution for  $\gamma_0$  sufficiently small, which moreover has conical singularities along  $D$ . It should be pointed out that the Kähler-Einstein metric  $\omega_\gamma$  on  $X - D$  produced in the proof of Theorem 1.8, a priori, only has a volume form with conical singularities along  $D$ . However, in the orbifold case, i.e. when  $\gamma = 1 - 1/m$  for some positive

integer  $m$ , it follows from standard arguments that the metric  $\omega_\gamma$  indeed has cone singularities (see the discussion below for the general case). Donaldson proposed another approach to solving the equation 1.10 for  $\gamma$  sufficiently small by perturbing the complete Ricci flat metric of Tian-Yau on  $X - D$  which, at least formally, is a solution of equation 1.10 when  $\gamma = 0$  [89]. This can be seen as a singular variant of the usual continuity method. One virtue of the present approach is thus that it bypasses the openness problem in the proposed continuity method.

It should be pointed out that in case of negative Ricci curvature the existence of Kähler-Einstein metric with conical singularities along a divisor was previously conjectured by Tian [87] in connection to applications to algebraic geometry and further studied by Jeffres [54, 55] and Mazzeo [67] (where an existence result was announced for  $\gamma \in ]0, 1/2[$ ).

After the first version of the present paper appeared on ArXiv there have been several important new developments concerning Kähler-Einstein metrics with conical singularities along a divisor that we next briefly describe, referring to the cited papers for precise statements. In the paper [45] Donaldson established the openness property with respect to the strictly positive parameter  $\gamma$  of solutions to equation 1.10 with certain further regularity properties (defined using weighted Hölder spaces adapted to  $D$ ). Using Donaldson's result and a perturbation trick in [54, 55, 67] Brendle [19] proved the existence of Ricci flat metrics with conical singularities along a given divisor, assuming that  $\gamma \in ]0, 1/2[$ .<sup>2</sup> A very general existence and regularity theory for Kähler-Einstein with conical singularities along a divisor (or in other words Kähler-Einstein *edge metrics*) has been developed by Jeffres-Mazzeo-Rubinstein [56] based on the edge calculus [66] combined with a continuity method. In particular, in the positively curved case, the results in [56] say that if the twisted Mabuchi functional corresponding to a pair  $(X, (1 - \gamma)D)$  is proper then there is a Kähler-Einstein metric with appropriate cone singularities and a complete asymptotic expansion along  $D$ , only assuming that  $\gamma \in ]0, 1[$  (we refer to [56] for the precise regularity statement and the corresponding function spaces). Since we have shown that the properness does hold for  $\gamma < \Gamma$  in Donaldson's setting, i.e. when  $D$  is an anti-canonical divisor, the results in [56] hence imply that the solutions in  $\omega_\gamma$  in Theorem 1.8 indeed always have conical singularities and moreover admit a complete asymptotic expansion. In another direction Campana-Guenancia-Păun [22] used a direct regularization argument to produce negatively curved Kähler-Einstein metrics with cone singularities along a given klt divisor  $\Delta = \sum_i (1 - \gamma_i)D_i$ , assuming  $\gamma_i \in ]0, 1/2[$ .

**Organization.** In Section 2 we setup the pluripotential theoretic and functional analytical framework, emphasizing the role of Legendre transforms (in infinite dimensions). In section 3 the main results concerning general Monge-Ampère mean field equations stated in the introduction are proved. In the following sections these results are applied and refined in the setting of twisted Kähler-Einstein metrics (section 4) and log Fano manifolds and Donaldson's equation (section 5). In the appendix we generalize some results of Demailly on the relation between  $\alpha$ -invariants and log canonical thresholds to the setting of klt pairs.

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<sup>2</sup>Combining the arguments in [45, 19] with those in the present paper the author then noted that  $\omega_\gamma$  has conical singularities for any  $\gamma$  sufficiently small thus confirming Donaldson's conjecture (see arXiv:1011.3976 [v3]). More precisely the result was shown to hold for  $\gamma < \min\{\Gamma, 1/2\}$  by deforming any orbifold solution. Here we have omitted the argument as the subsequent results [56] permit to remove the unnatural restriction  $\gamma < 1/2$  (as explained above).

**Acknowledgments.** I am very grateful to Sébastien Boucksom, Vincent Guedj and Ahmed Zeriahi for the stimulating collaboration [10] which paved the way for the present paper. Also thanks to Bo Berndtsson for discussions related to [13], to Yanir Rubinstein for once sending me his thesis where I learned about the  $C^2$ -estimate in [4, 73] and Valentino Tosatti and Gabor Székelyhidi for comments.

**Notational remark.** Throughout,  $C, C'$  etc denote constants whose values may change from line to line

## 2. FUNCTIONALS ON THE SPACES OF PROBABILITY MEASURES AND $\omega$ -PSH FUNCTIONS AND LEGENDRE DUALITY

In this section we will consider various functionals defined on the space  $\mathcal{M}_1(X)$  of probability measures on  $X$ , as well as on the space  $PSH(X, \omega)$  of  $\omega$ -*psh functions* on  $X$  (also called *potentials*). The definition of the space  $PSH(X, \omega)$  will be recalled below. It will be important to also work with different subspaces of these spaces:

$$\mathcal{H}(X, \omega) := \begin{array}{ccccc} \{Volume\ forms\} & \subset & E_1(X, \omega) & \subset & \mathcal{M}_1(X) \\ \{K\ddot{a}hler\ potentials\} & \subset & \mathcal{E}^1(X, \omega) & \subset & PSH(X, \omega) \end{array}$$

where  $E_1(X, \omega)$  and  $\mathcal{E}^1(X, \omega)$  are the subspaces of *finite energy* elements. These notions are higher dimensional versions of the energy notions familiar from the classical theory of Dirichlet spaces on Riemann surfaces. The general definitions and relations to Legendre transforms will be recalled below. The point will be that  $\mathcal{E}^1(X, \omega)$  is the space where a certain concave functional  $\mathcal{E}_\omega$  on  $PSH(X, \omega)$  is finite, while  $E_1(X, \omega)$  is the space where the “restricted” Legendre transform of  $\mathcal{E}_\omega$  is finite. The subspaces are then mapped to each others under the differential  $d\mathcal{E}_\omega$  (and its inverse). In fact, the differential  $d\mathcal{E}_\omega$  at  $u$  is nothing but the complex Monge-Ampère measure of  $u$ .

We start with the following

**2.1. Functional analytic framework and Legendre-Fenchel transforms.** We equip the space  $\mathcal{M}(X)$  of all signed finite Borel measures on  $X$  with its the usual weak topology, i.e.  $\mu_j \rightarrow \mu$  iff

$$\langle u, \mu_j \rangle := \int_X u \mu_j \rightarrow \int_X u \mu$$

for any continuous function  $u$ , i.e. for all  $u \in C^0(X)$ . In other words,  $\mathcal{M}(X)$  is the topological dual of the vector space  $C^0(X)$ . We will be mainly concerned with the subspace  $\mathcal{M}_1(X)$  of all probability measures on  $X$  which is a convex compact subset of  $\mathcal{M}(X)$ . This latter space is a locally convex topological vector space [33]. As such it admits a good duality theory[33]: given a functional  $\Lambda$  on the vector space  $C^0(X)$  its *Legendre(-Fenchel) transform* is the following functional  $\Lambda^*$  on  $\mathcal{M}(X)$ :

$$\Lambda^*(\mu) := \sup_{u \in C^0(X)} (\Lambda(u) - \langle u, \mu \rangle)$$

Conversely, if  $H$  is a functional on the vector space  $\mathcal{M}(X)$  we let

$$H^*(u) := \inf_{\mu \in \mathcal{M}(X)} (H(\mu) + \langle u, \mu \rangle)$$

Note that we are using rather non-standard sign conventions. In particular,  $\Lambda^*(\mu)$  is always *convex* and *lower semi-continuous (lsc)*, while  $H^*(u)$  is *concave* and *upper-semicontinuous (usc)*. As a well-known consequence of the Hahn-Banach separation

theorem we have the following fundamental duality relation [33]:

$$(2.1) \quad \Lambda = (\Lambda^*)^*$$

iff  $\Lambda$  is concave and usc. We also recall the following basic fact (we will not use the uniqueness property, only the minimization property)

**Lemma 2.1.** *Assume that  $\Lambda$  is a functional on  $C^0(X)$  which is finite, concave and Gateaux differentiable (i.e differentiable along lines). Then, for a fixed  $u \in C^0(X)$  the differential  $d\Lambda|_u$  is the unique minimizer of the following functional on  $\mathcal{M}(X)$  :*

$$(2.2) \quad \mu \mapsto \Lambda^*(\mu) + \langle u, \mu \rangle$$

(and the minimum value equals  $\Lambda(u)$ ).

*Proof.* As a courtesy to the reader we give the simple proof. By the duality relation 2.1 the minimal value of the functional 2.2 is indeed  $\Lambda(u)$ , which means that  $\mu_u$  is a minimizer iff

$$\Lambda(u) \leq \Lambda(u') + \langle u - u', \mu_u \rangle$$

for all  $u' \in C^0(X)$ . When  $\mu = d\Lambda|_u$  the previous inequality follows immediately from the concavity of  $\Lambda$ . More generally, any  $\mu_u$  satisfying the previous inequality is called a *subdifferential* for  $\Lambda$  at  $u$ . To prove uniqueness we take  $u' = u + tv$  for  $v \in C^0(X)$  and  $t \in \mathbb{R}$  and divide the previous inequality by  $t$ , letting  $t$  tend to zero, first for  $t > 0$  and then for  $t < 0$ , giving

$$\frac{d\Lambda(u + tv)}{dt} \Big|_{t=0^-} \leq \langle v, \mu_u \rangle \leq \frac{d\Lambda(u + tv)}{dt} \Big|_{t=0^+}$$

Since  $\Lambda$  is Gateaux differentiable the left and right derivative above coincide forcing  $\langle v, \mu_u \rangle = \langle v, d\Lambda|_u \rangle$  for any  $v \in C^0(X)$ .  $\square$

Conversely, if the functional in the statement of the lemma above has a unique maximizer  $\mu_u$  then  $\Lambda$  is Gateaux differentiable with  $d\Lambda|_u = \mu_u$ . We will prove a variant of this fact in Prop 2.7 below.

**2.2. The space  $PSH(X, \omega)$  of  $\omega$ -psh functions.** A general reference for this section is [51]. The space  $PSH(X, \omega)$  of  $\omega$ -psh functions (sometimes simple called *potentials*) is defined as the space of all functions  $u \in L^1(X) (= L^1(X, \omega^n))$  with values in  $[-\infty, \infty[$  which are upper semi-continuous and such that

$$\omega_u := \omega + dd^c u \geq 0$$

in the sense of currents. We endow  $PSH(X, \omega)$  with the  $L^1$ -topology. There is a basic continuous bijection [51]

$$u \mapsto \omega_u, \quad PSH(X, \omega)/\mathbb{R} \leftrightarrow \{\text{positive closed currents in } [\omega]\}$$

where the right hand side is equipped with the weak topology (and the space coincides with  $\mathcal{M}_1(X)$  when  $n = 1$  and  $V = 1$ ). In particular, this shows that  $PSH(X, \omega)/\mathbb{R}$  is *compact*. The subspace of all *Kähler potentials* is defined by

$$\mathcal{H}(X, \omega) := \{u \in C^\infty(X) : \omega_u > 0\}$$

so that  $\mathcal{H}(X, \omega)/\mathbb{R}$  is isomorphic to the space of all Kähler forms in the class  $[\omega]$ . By the fundamental approximations results of Demailly [34]  $\mathcal{H}(X, \omega)$  is dense in  $PSH(X, \omega)$ . See also [17] for a simple proof of the last statement in the following proposition.

**Proposition 2.2.** *The space  $\mathcal{H}(X, \omega)$  is dense in  $PSH(X, \omega)$  (wrt the  $L^1$ -topology):*

$$PSH(X, \omega) = \overline{\mathcal{H}(X, \omega)}$$

*More precisely, any  $\omega$ -psh function can be written as a decreasing limit of elements  $u_j$  in  $\mathcal{H}(X, \omega)$ .*

**2.3. The Monge-Ampère operator and the functional  $\mathcal{E}_\omega(u)$ .** In this section and the following one we recall notions and results from [52, 14, 10] (a part from Prop 2.7, which is new). Let us start by recalling the definition of the Monge-Ampère measure  $MA(u)$  on *smooth* functions. It is defined by

$$MA(u) := \frac{(\omega + dd^c u)^n}{Vn!} =: \frac{(\omega_u)^n}{Vn!}$$

which is hence a (positive) probability measure when  $u \in PSH(X, \omega)$ . The Monge-Ampère  $MA$  operator may be naturally identified with a one-form on the vector space  $C^\infty(X)$  by letting

$$\langle MA|_u, v \rangle := \int_X MA(u)v$$

for  $u \in C^\infty(X)$ . As observed by Mabuchi [64, 65] (in the context of Kähler-Einstein geometry) the one-form  $MA$  is closed and hence it has a primitive  $\mathcal{E}_\omega$  (defined up to an additive constant) on the space all smooth weights, i.e.

$$(2.3) \quad d\mathcal{E}|_u = MA(u)$$

We fix the additive constant by requiring  $\mathcal{E}_\omega(0) = 0$ . Integrating  $\mathcal{E}_\omega$  along line segments one arrives at the following well-known formula:

$$(2.4) \quad \mathcal{E}_\omega(u) := \frac{1}{(n+1)!V} \sum_{j=0}^n \int_X \omega_u^j \wedge (\omega)^{n-j}$$

Conversely, one can simply take this latter formula as the definition of  $\mathcal{E}_\omega$  and observe that the following proposition holds (compare [9] for a more general singular setting):

**Proposition 2.3.** *The following holds*

(i) *The differential of the functional  $\mathcal{E}_\omega$  at a smooth function  $u$  is represented by the measure  $MA(u)$ , i.e.*

$$(2.5) \quad \frac{d}{dt}_{t=0} (\mathcal{E}_\omega(u + tv)) = \int_X MA(u)v$$

(ii)  *$\mathcal{E}_\omega$  is increasing on the space of all smooth  $\omega$ -psh functions*

(iii)  *$\mathcal{E}_\omega$  is concave on the space of all smooth  $\omega$ -psh functions and when  $n = 1$  it is concave on all of  $C^\infty(X)$*

Note that (ii) is a direct consequence of (i), since the differential of  $\mathcal{E}_\omega$  is represented by a (positive) measure.

**2.3.1. The general singular setting.** One first extends the functional  $\mathcal{E}_\omega$  (formula 2.4) to all  $\omega$ -psh functions by defining

$$\mathcal{E}_\omega(u) := \inf_{u' \geq u} \mathcal{E}_\omega(u') \in [-\infty, \infty[$$

where  $u$  ranges over all locally bounded (or smooth)  $\omega$ -psh functions  $u'$  such that  $u' \geq u$ . Next, we let

$$\mathcal{E}^1(X, \omega) := \{u \in PSH(X, \omega) : \mathcal{E}_\omega(u) > -\infty\},$$

that we will refer to as the space of all  $\omega$ -psh functions with *finite (pluri-)energy*. In the Riemann surface case  $\mathcal{E}^1(X, \omega)$  is the classical Dirichlet subspace of  $PSH(X, \omega)$  consisting of all functions whose gradient is in  $L^2(X)$ .

As a consequence of the monotonicity of  $\mathcal{E}_\omega(u)$  and Bedford-Taylor's fundamental local continuity result for mixed Monge-Ampère operators one obtains the following proposition (cf. [14], Prop 2.10; note that  $\mathcal{E}_\omega = -E_\chi$  for  $\chi(t) = t$  in the notation in op. cit.)

**Proposition 2.4.** *The functional  $\mathcal{E}_\omega(u)$  is upper semi-continuous on  $PSH(X, \omega)$ , concave and non-decreasing. Moreover, it is continuous wrt decreasing sequences in  $PSH(X, \omega)$ .*

For any  $u \in \mathcal{E}^1(X, \omega)$  the (non-pluripolar) Monge-Ampère measure  $MA(u)$  is well-defined [14] and does not charge any pluripolar sets. We collect the continuity properties that we will use in the following [14]

**Proposition 2.5.** *Let  $(u^{(i)}) \subset \mathcal{E}^1(X, \omega)$  be a sequence decreasing to  $u \in \mathcal{E}^1(X, \omega)$ . Then, as  $i \rightarrow \infty$ ,*

$$MA(u_i) \rightarrow MA(u)$$

and

$$u_i MA(u_i) \rightarrow u MA(u)$$

in the weak topology of measures and  $\mathcal{E}_\omega(u_j) \rightarrow \mathcal{E}_\omega(u)$ .

In particular, by the previous proposition we could as well have defined  $MA(u)$  for  $u \in \mathcal{E}^1(X, \omega)$  as the limit of the volume forms  $MA(u_j)$  with  $u_j \in \mathcal{H}(X, \omega)$  any sequence decreasing to  $u$  (using Demailly's approximation result [34]).

**2.4. The pluricomplex energy  $E(\mu)$  and potentials.** Following [14] we define the (pluricomplex) energy by

$$(2.6) \quad E(\mu) := \sup_{u \in PSH(X, \omega)} \mathcal{E}_\omega(u) - \langle u, \mu \rangle$$

if  $\mu \in \mathcal{M}_1(X)$ . It will also be useful to extend  $E$  to all of the vector space  $\mathcal{M}(X)$  by letting  $E(\mu) = \infty$  on  $\mathcal{M}(X) - \mathcal{M}_1(X)$ . We will denote the subspace of all finite energy probability measures by

$$E_1(X, \omega) := \{\mu : E(\mu) < \infty\}$$

By Propositions 2.5 and 2.2 it is enough to take the sup over the subspace  $C^0(X) \cap PSH(X, \omega)$  or even over the subspace  $\mathcal{H}(X, \omega)$  of Kähler potentials. But one point of working with less regular functions is that the sup can be attained. Indeed, as recalled in the following theorem

$$(2.7) \quad E(\mu) := \mathcal{E}_\omega(u_\mu) - \langle u_\mu, \mu \rangle$$

for a unique potential  $u_\mu \in \mathcal{E}^1(X, \omega)/\mathbb{R}$  of  $\mu$  if  $E(\mu) < \infty$  where

$$(2.8) \quad MA(u_\mu) = \mu.$$

(the uniqueness result for bounded potentials was first shown by Blocki [15])

**Theorem 2.6.** [10] *The following is equivalent for a probability measure  $\mu$  on  $X$  :*

- $E(\mu) < \infty$
- $\langle u, \mu \rangle < \infty$  for all  $u \in \mathcal{E}^1(X, \omega)$
- $\mu$  has a potential  $u_\mu \in \mathcal{E}(X, \omega)$ , i.e. equation 2.8 holds

Moreover,  $u_\mu$  is uniquely determined mod  $\mathbb{R}$ , i.e. up to an additive constant and can be characterized as the function maximizing the functional whose sup defines  $E(\mu)$  (formula 2.6). More generally, if  $u_j$  is a sequence in  $\mathcal{E}^1(X, \omega)$  such that  $\sup_X u_j = 0$  and

$$\liminf_j \mathcal{E}_\omega(u_j) - \langle u_j, \mu \rangle \geq E(\mu)$$

then  $u_j \rightarrow u_\mu$  where  $u_\mu$  is the unique potential of  $\mu$  such that  $\sup_X u_\mu = 0$

The previous theorem was proved in [10] using the variational approach in the more general setting of a big class  $[\omega]$ . In the case when  $\mu$  is a volume form Yau's seminal theorem [93] furnishes a *smooth* potential  $u_\mu$  (using the continuity method).

We will next prove a dual version of 2.3 which is a new result in the general non-smooth setting. If the functional  $\langle \mu, \cdot \rangle$  were lsc on all of  $\mathcal{E}^1(X, \omega)$  then the proposition below would essentially be a consequence of the extremal property of  $u_\mu$  given by Theorem 2.6 combined with a dual version of the converse of Lemma 2.1 on  $\mathcal{M}(X)$ .

**Proposition 2.7.** *Let  $\mu^t = \mu^0 + t\nu$  be a segment in  $E_1(X, \omega) := \{E < \infty\}$  where  $t \in ]-\epsilon, \epsilon[$  for some  $\epsilon > 0$ . Then*

$$(2.9) \quad \frac{dE(\mu^t)}{dt} \Big|_{t=t_0} = - \int_X u_{\mu^{t_0}} \nu,$$

where  $u_{\mu^t}$  is the potential of  $\mu$  (which is unique mod  $\mathbb{R}$ ). Moreover, for any two elements  $\mu^1$  and  $\mu^0$  of  $E_1(X, \omega)$  we have

$$(2.10) \quad E(\mu^1) \geq E(\mu^0) + \int_X (-u_{\mu^0})(\mu^1 - \mu^0),$$

*Proof.* Denote by  $u^t$  the potential of  $\mu^t$  normalized so that  $\sup u^t = 0$ . Then

$$(2.11) \quad \frac{1}{t} E((\mu^t) - E(\mu^0)) = \frac{1}{t} ((\mathcal{E}_\omega(u^t) - \langle u^t, \mu^0 \rangle) - (\mathcal{E}_\omega(u^0) - \langle u^0, \mu^0 \rangle)) - \langle u^t - u^0, \nu \rangle + \langle -u^0, \nu \rangle$$

*Step one:*  $\langle u^t - u^0, \nu \rangle \rightarrow 0$  as  $t \rightarrow 0$ .

First observe that there is a constant  $C$  such that

$$\text{Claim 1: } u^t \in \{\mathcal{E}_\omega \geq -C\} \cap \{\sup_X = 0\}$$

Indeed, by the extremal property of  $u^t$  we have  $\mathcal{E}_\omega(u^t) - (\langle u^t, \mu^0 \rangle + t \langle u^t, \nu \rangle) =$

$$\mathcal{E}_\omega(u^t) - \langle u^t, \mu^t \rangle \geq \mathcal{E}_\omega(u^0) - \langle u^0, \mu^t \rangle = C - t \langle u^0, \nu \rangle \geq C''$$

Moreover, as shown in [10] (Prop 3.4), for any  $\mu \in E_1(X, \omega)$  there is a constant  $C_\mu$  such that

$$(2.12) \quad |\langle u^t, \mu \rangle| \leq C_\mu (-\mathcal{E}(u^t))^{1/2}$$

if  $u^t \in \{\mathcal{E}_\omega > -\infty\} \cap \sup = 0$ . Combining this latter inequality with the previous ones gives

$$\mathcal{E}_\omega(u^t) \geq -C'' - C'''(1+t)(-\mathcal{E}(u^t))^{1/2}$$

which proves the claim (since  $t$  is bounded).

Next, we will prove the following

$$\text{Claim 2: } \liminf_{t \rightarrow 0} \mathcal{E}_\omega(u^t) - \langle u^t, \mu^0 \rangle \geq \mathcal{E}_\omega(u^0) - \langle u^0, \mu^0 \rangle$$

As above, by the extremal property of  $u^t$  it is enough to prove that

$$\langle u^t, \mu^t \rangle - \langle u^t, \mu^0 \rangle = t \langle u^t, \nu \rangle \rightarrow 0$$



as  $t \rightarrow 0$ . But this follows from the upper bound 2.12 combined with claim 1 above.

Now, Claim 2 combined with the last statement in Theorem 2.6 shows that  $u^t \rightarrow u^0$  in  $L^1(X, \omega^n)$  when  $t \rightarrow 0$ . As shown in [10] for any  $\mu \in E_1(X, \omega)$  (and trivially also for the difference  $\nu$  of elements in  $E_1(X, \omega)$ ) the functional  $\langle \cdot, \mu \rangle$  is continuous wrt the  $L^1$ -topology on the subset in the Claim 1. This finishes the proof of step one.

*Step two: proof of formula 2.9*

By concavity the function of  $t$  inside the first bracket in the rhs of 2.11 achieves its maximum on  $] - \epsilon, \epsilon[$  at the value  $t = 0$  and hence letting  $t \rightarrow 0^+$  gives

$$\frac{dE(\mu^t)}{dt} \Big|_{t=0^+} \leq 0 + 0 - \langle -u^0, \nu \rangle$$

Similarly,

$$\frac{dE(\mu^t)}{dt} \Big|_{t=0^-} \geq 0 + 0 - \langle -u^0, \nu \rangle$$

But by the convexity of  $E(\mu^t)$  we have  $\frac{dE(\mu^t)}{dt} \Big|_{t=0^-} \leq \frac{dE(\mu^t)}{dt} \Big|_{t=0^+}$  which finally proves the equality 2.9.

*Step three: proof of inequality 2.10*

Let now  $\mu_t$  be the affine segment, with  $t \geq 0$ , connecting the given points  $\mu^0$  and  $\mu^1$ . Combining the convexity of  $E(\mu^t)$  and formula 2.9 (evaluated at  $t = t_0 > 0$ ) we have

$$E(\mu^1) \geq E(\mu^{t_0}) + \int_X (-u_{\mu^{t_0}})(\mu^1 - \mu^0)(1 - t_0)$$

and hence letting  $t_0 \rightarrow 0$  and using step one above and the fact that  $E$  is lower semi-continuous finishes the proof of the proposition.  $\square$

Note that since the integral of  $\nu$  vanishes the derivative above is independent of the normalization of  $u_\mu$ .

Before continuing we note that  $E(\mu)$  is *not* (at least as it stands) a Legendre transform of  $\mathcal{E}_\omega(u)$  even when restricted to  $\mathcal{M}_1(X)$ , because as explained above the sup must be taken over the convex subspace  $C^0(X) \cap PSH(X, \omega)$  of the vector space  $C^0(X)$ . In order to realize  $E$  as a Legendre transform we turn to the definition of the projection operator  $P_\omega$ .

*Remark 2.8.* When  $n = 1$  the sup referred to above may actually be taken over all of  $C^0(X)$ . Indeed, as explained above the extremizer  $u_\mu$  a posteriori satisfies  $\omega_{u_\mu} = \mu \geq 0$  and hence  $E$  is indeed the Legendre transform of  $\mathcal{E}^*$  in the Riemann surface case.

**2.5. The psh projection  $P$  and the formula  $E = (\mathcal{E} \circ P)^*$ .** Consider the following projection operator  $P_\omega : C^0(X) \rightarrow C^0(X) \cap PSH(X, \omega)$

$$P_\omega u := \sup \{v(x) : v \in PSH(X, \omega), v \leq u \text{ on } X\}$$

(the lower semi-continuity of  $P_\omega u$  follows from Demailly's approximation result which allows us to write  $P_\omega u$  as an upper envelope of continuous functions and the upper semi-continuity is obtained by noting that  $P_\omega u$  is a candidate for the sup in its definition). One of the main results in [9] is the following

**Theorem 2.9.** (*B.-Boucksom [9]*) *The functional  $\mathcal{E}_\omega \circ P_\omega$  is concave and Gateaux differentiable on  $C^0(X)$ . More precisely,*

$$d(\mathcal{E}_\omega \circ P_\omega)|_u = MA(\mathcal{E}_\omega(P_\omega u))$$

The differentiability of the composed map  $\mathcal{E}_\omega \circ P_\omega$  should be contrasted with the fact that the non-linear projection  $P_\omega$  is certainly not differentiable. The main ingredient in the proof of the previous theorem is the following orthogonality relation:

$$(2.13) \quad \langle MA(Pu), (u - Pu) \rangle = 0,$$

Note that it follows immediately from the fact that  $Pu \leq u$  that

$$(2.14) \quad E = (\mathcal{E}_\omega \circ P_\omega)^* \text{ on } \mathcal{M}_1(X)$$

Moreover, by the previous theorem

$$(2.15) \quad d(\mathcal{E}_\omega \circ P_\omega)(C^0(X)) \subset \mathcal{M}_1(X) \subset \mathcal{M}(X)$$

These two facts together give the following proposition, saying that the functionals  $E$  and  $\mathcal{E}_\omega \circ P_\omega$  are Legendre transformations of each others (giving a slightly more precise version of Theorem 5.3 in [10] when  $K = X$  there)

**Proposition 2.10.** *The relation 2.14 holds on all of the vector space  $\mathcal{M}(X)$  of signed measures on  $X$ , i.e.*

$$E = (\mathcal{E}_\omega \circ P_\omega)^* \text{ on } \mathcal{M}(X)$$

and dually

$$\mathcal{E}_\omega \circ P_\omega = E^* \text{ on } \mathcal{C}^0(X)$$

*Proof.* Since by definition  $E = \infty$  on  $\mathcal{M}(X) - \mathcal{M}_1(X)$  we have for any  $u \in C^0(X)$

$$E^*(u) := \inf_{\mu \in \mathcal{M}(X)} (E(\mu) + \langle u, \mu \rangle) = \inf_{\mu \in \mathcal{M}_1(X)} (E(\mu) + \langle u, \mu \rangle),$$

and hence the identity 2.14 combined with 2.15 and Lemma 2.1 (not using the uniqueness) gives, with  $\Lambda := \mathcal{E} \circ P$ ,

$$E^*(u) := \inf_{\mu \in \mathcal{M}_1(X)} (\Lambda^*(\mu) + \langle u, \mu \rangle) = \inf_{\mu \in \mathcal{M}(X)} (\Lambda^*(\mu) + \langle u, \mu \rangle)$$

Finally, by the duality relation 2.1 this means that  $E^*(u) = (\Lambda^*)$  and applying the Legendre transform again also gives  $E = \Lambda^*$ .  $\square$

In particular, it follows immediately from the previous proposition that

$$E^* = \mathcal{E}_\omega \text{ on } \mathcal{C}^0(X) \cap PSH(X, \omega)$$

**2.6. The relative entropy  $D(\mu)$  and its Legendre transform  $\mathcal{L}^-$ .** The relative entropy  $D_{\mu_0}(\mu) := D(\mu)$  wrt a fixed probability measure  $\mu_0$  is defined by

$$D(\mu) := \int_X \log(\mu/\mu_0) \mu$$

when  $\mu$  is absolutely continuous wrt  $\mu_0$  and otherwise  $D(\mu) := \infty$ . As is well-known (see for example [33])  $D$  is the Legendre transform, i.e.  $D = \mathcal{L}^*$ , of the following functional on  $C^0(X)$  :

$$\mathcal{L}_{\mu_0}(u) := -\log \int_X e^u \mu_0$$

(compare the proof of Lemma 2.12). More generally, for any given parameter  $\beta \in \mathbb{R} - \{0\}$  and measurable function  $u$ ,

$$\mathcal{L}_{\mu_0, \beta}(u) := -\frac{1}{\beta} \log \int_X e^{\beta u} \mu_0$$

which in particular defines a functional on  $C^0(X)$  which, by Hölder's inequality is concave for  $\beta > 0$  and convex for  $\beta < 0$ . The following basic duality relation holds when  $\beta > 0$  [33]

$$\mathcal{L}_\beta^*(-\mu) = \frac{1}{\beta} D(\mu)$$

i.e.

$$\frac{1}{\beta} D(\mu) = \sup_{u \in C^0(X)} \left( -\frac{1}{\beta} \log \int_X e^{\beta u} \mu_0 + \langle u, \mu \rangle \right)$$

Similarly, if  $\beta = -\gamma$  with  $\gamma > 0$  then we have that

$$\mathcal{L}_\gamma^-(u) := -\mathcal{L}_{\mu_0, -\gamma}(u) := -\frac{1}{\gamma} \log \int_X e^{-\gamma u} \mu_0$$

is a concave functional and by symmetry

$$\mathcal{L}_\gamma^{-*} = \frac{1}{\gamma} D$$

i.e.

$$\frac{1}{\gamma} D(\mu) = \sup_{u \in C^0(X)} \left( -\frac{1}{\gamma} \log \int_X e^{-\gamma u} \mu_0 - \langle u, \mu \rangle \right)$$

Note that on  $C^0(X)$  it follows directly from the chain rule that

$$d\mathcal{L}_\gamma^- = \frac{e^{-\gamma u} \mu_0}{\int_X e^{-\gamma u} \mu_0}$$

so that the image of  $C^0(X)$  under  $d\mathcal{L}_\gamma^-$  is the subspace of  $\mathcal{M}_1(X)$  of all measures  $\mu$  with strictly positive continuous density wrt  $\mu_0$ . However we will need to calculate the derivatives with almost no regularity assumptions.

**Proposition 2.11.** *Let  $\mu^t = \mu^0 + t\nu$  be a segment in  $\{D < \infty\}$ . Then*

$$\frac{dD(\mu^t)}{dt} \Big|_{t=0^+} = \int_X \log(\mu^0/\mu_0) \nu$$

*if the right hand side above is finite. Similarly, let  $u^t = u + t\nu$  be a segment in the space of all usc functions where  $\mathcal{L}_\gamma^-(u)$  is finite. Then*

$$\frac{d\mathcal{L}_\gamma^-(u^t)}{dt} \Big|_{t=0^+} = \int_X \frac{\nu e^{-\gamma u} \mu_0}{\int_X e^{-\gamma u} \mu_0}$$

*if the right hand side above is finite.*

*Proof.* By definition

$$\frac{1}{t} (D(\mu^t) - D(\mu^0)) = \int_X \frac{1}{t} (\log(\frac{\mu^t}{\mu_0}) - \log(\frac{\mu^0}{\mu_0})) \mu^0 + \int_X \log(\frac{\mu^t}{\mu_0}) \nu$$

Since  $x \mapsto \log x$  is monotone and convex with derivative  $1/x$  when  $x > 0$  the integrands above are monotone in  $t$  and hence the monotone convergence theorem gives

$$\frac{dD(\mu^t)}{dt} \Big|_{t=0^+} = \int_X \frac{\nu}{\mu^0} \mu^0 + \int_X \log(\frac{\mu^0}{\mu_0}) \nu$$

By assumption  $\int_X \nu = 0$  and hence the first term above vanishes which proves the first formula in the proposition.

The second formula of the theorem is proved in a similar fashion now using that  $x \mapsto e^x$  is convex (exactly as in the proof of Lemma 6.1 in [10])  $\square$

Now we can prove the following

**Lemma 2.12.** *Let  $\mu$  be a finite energy measure and assume that  $u \in \mathcal{E}^1(X, \omega)$  with  $\int_X e^{-\gamma u} \mu_0 < \infty$ . Then*

$$(2.16) \quad (\mathcal{L}_\gamma^-)^*(\mu) = \log\left(-\frac{1}{\gamma} \int_X e^{-\gamma u} \mu_0\right) - \langle u, \mu \rangle \quad (:= \mathcal{N}(u))$$

*iff*

$$(2.17) \quad \mu = \frac{e^{-\gamma u} \mu_0}{\int_X e^{-\gamma u} \mu_0}$$

*Proof.* First note that by the assumptions on  $u$  and  $\mu$  both terms in the definition of  $\mathcal{N}(u)$  above are finite. Assume first that  $u$  satisfies 2.17. If  $v$  denotes a fixed continuous function on  $X$  and  $u_t := u + tv$ , then according to the previous proposition

$$(2.18) \quad \frac{d(\mathcal{N}(u_t))}{dt} \Big|_{t=0^+} = 0$$

By concavity it follows that  $\mathcal{N}(u) \geq \mathcal{N}(u + tv)$  for any  $t \geq 0$  and in particular for  $t = 1$ . Now take an arbitrary function  $w \in C^0(X)$  and write the lsc function  $w - u$  as an increasing limit of continuous functions  $v_j$ . Since, as explained above,

$$\mathcal{N}(u) \geq \mathcal{N}(u + v_j)$$

letting  $j \rightarrow \infty$  and using the monotone convergence theorem gives

$$\mathcal{N}(u) \geq \sup_{w \in C^0(X)} \mathcal{N}(w) := (\mathcal{L}_\gamma^-)^*(\mu)$$

Similarly, writing  $u$  as a decreasing limit of continuous functions  $w_j$  and passing to the limit forces equality above.

Conversely, assume that  $u$  satisfies 2.16 above. Then it follows in particular (approximating as above) that the differentiable function

$$t \mapsto \mathcal{N}(u_t)$$

with  $u_t$  as above attains its maximum at  $t = 0$ . Hence, the critical point equation 2.18 holds and since  $v$  was arbitrary it follows by the formula in the previous proposition that  $u$  satisfies the relation 2.17.  $\square$

**2.7. Properness and coercivity of functionals.** The energy functional  $E$  defines an exhaustion function on the space  $E_1(X, \omega)$  (i.e. the sets  $\{E \geq -C\}$  are compact, since  $E$  is lsc, and their union is  $E_1(X, \omega)$ ). A functional  $F(\mu)$  on  $E_1(X, \omega)$  is said to be *proper (wrt energy)* if it is proper with respect to the previous exhaustion, i.e.

$$E(\mu) \rightarrow \infty \implies F(\mu) \rightarrow \infty$$

and *coercive* (which is a stronger condition) if there are positive constants  $a$  and  $b$  such that

$$F \geq aE - b$$

Similarly, the functional  $-\mathcal{E}_\omega$  defines an exhaustion function on the space  $\mathcal{E}^1(X, \omega)$  (it is indeed lsc according to 2.4). To get an exhaustion function of  $\mathcal{E}^1(X, \omega)/\mathbb{R}$  one replaces  $-\mathcal{E}_\omega$  with its  $\mathbb{R}$ -invariant analogue

$$J_\omega(u) := -\mathcal{E}_\omega(u) + \int_X u \frac{\omega^n}{Vn!}$$

often called Aubin's  $J$ -functional in the Kähler geometry literature. This then gives a notion of properness (wrt energy) and coercivity on  $\mathcal{E}^1(X, \omega)/\mathbb{R}$ , as well, introduced by Tian in the setting of Kähler geometry (see [86] and references therein)

In fact, the notions of properness and coercivity above are preserved under the bijection

$$\mathcal{E}^1(X, \omega)/\mathbb{R} \rightarrow E_1(X, \omega) : u \mapsto MA(u)$$

as follows from the following basic lemma, which also involves Aubin's  $I$ -functional:

$$I_\omega(u) := -\frac{1}{Vn!} \int u(\omega_u^n - \omega^n)$$

**Lemma 2.13.** *The following identity holds*

$$E(MA(u)) = (I_\omega - J_\omega)(u)$$

and the following inequality

$$(2.19) \quad \frac{1}{n} J_\omega \leq (I_\omega - J_\omega) \leq n J_\omega$$

*Proof.* First observe that the following identity holds:

$$E(MA(u)) = (I_\omega - J_\omega)(u),$$

Indeed, by 2.7,

$$\begin{aligned} E(MA(u)) &= \mathcal{E}_\omega(u) - \langle u, MA(u) \rangle = \\ &= (\mathcal{E}_\omega(u) - \langle u, MA(0) \rangle) + \langle -u, MA(u) - MA(0) \rangle =: -J(u) + I(u) \end{aligned}$$

Hence, the lemma follows from the basic inequality 2.19 (see for example formula 2.7 in [10] for the non-regular case).  $\square$

In particular the previous lemma immediately gives the following inequality that we will have use for later: if  $\mu \in E_1(X, \omega)$  with potential  $u_\mu \in \mathcal{E}^1(X, \omega)$  normalized so that  $\int u_\mu \omega^n = 0$ , then

$$(2.20) \quad -\langle u_\mu, \mu \rangle \geq \left(\frac{n+1}{n}\right) E(\mu)$$

### 3. MONGE-AMPÈRE MEAN FIELD EQUATIONS AND MOSER-TRUDINGER TYPE INEQUALITIES

Fix a probability measure  $\mu_0$  of finite energy. Recall that  $\beta$  denotes a fixed parameter in  $\mathbb{R} - \{0\}$  and when  $\beta < 0$  we will often write  $\beta = -\gamma$ .

The (*normalized*) *Monge-Ampère mean field equation* (ME) associated to the triple  $(\omega, \mu_0, \beta)$  is the following equation for  $u \in \mathcal{E}^1(X, \omega)$

$$(3.1) \quad \frac{\omega_u^n}{Vn!} = \frac{e^{\beta u} \mu_0}{\int_X e^{\beta u} \mu_0}$$

where we recall that the measure in the left hand side above is the Monge-Ampère measure  $MA(u)$ . Thanks to the normalizing integral the equation is invariant under the additive action of  $\mathbb{R}$  on  $\mathcal{E}^1(X, \omega)$ . The *non-normalized* ME is the equation

$$(3.2) \quad \frac{\omega_u^n}{Vn!} = e^{\beta u} \mu_0$$

whose solutions are precisely the solutions of 3.1 with  $\int_X e^{\beta u} \mu_0 = 1$ . In general, the transformation  $u \mapsto u - \frac{1}{\beta} \log \int_X e^{\beta u} \mu_0$  clearly maps solutions of 3.1 to solutions of 3.2.

In this section we will be mainly concerned with the corresponding *free energy functional*

$$F_\beta(\mu) := E_\omega(\mu) + \frac{1}{\beta} \int_X \log\left(\frac{\mu}{\mu_0}\right) \mu$$

defined on the space  $E_1(X, \omega)$  of finite energy probability measures. We recall that the integral in the second term (i.e. the relative entropy) is by definition equal to  $\infty$  if  $\mu$  is not absolutely continuous wrt  $\mu_0$ . In particular,  $F_\beta(\mu)$  takes values in  $] -\infty, \infty]$  when  $\beta > 0$  and in  $[-\infty, \infty[$  when  $\beta < 0$ .

One of the reasons that we assume that  $\mu_0$  is of finite energy is that we will be interested in the cases when  $\beta < 0$  and the functional  $F_\beta$  admits a maximizer and in particular when it is bounded from above. But as shown below (see the first point of Theorem 3.4) a necessary condition for this is that  $\mu_0$  be of finite energy. We will also be interested in the closely related functional

$$\mathcal{G}_\beta(u) := \mathcal{E}_\omega(u) - \frac{1}{\beta} \log \int_X e^{\beta u} \mu_0 \in [-\infty, \infty[$$

defined on the space  $\mathcal{E}^1(X, \omega)$  of finite energy  $\omega$ -psh functions. To avoid notational complexity we will sometimes omit the subscripts  $\beta$ , (as well as the explicit dependence on  $\omega$  and  $\mu_0$ ).

We start with the following general regularity result whose first part is obtained by combining [58] and [79].

**Proposition 3.1.** *If  $\mu_0$  is a volume form then any solution  $u \in \mathcal{E}^1(X, \omega)$  to equation 3.2 is smooth. More generally, the solution is Hölder continuous under any of the following assumptions:*

- [59]  $\mu_0 = f dV$  where  $f \in L^p(X, dV)$  for some  $p > 1$  and where  $dV = \omega_0^n$  is the volume form on  $X$  of the metric  $\omega_0$ .
- [53]  $\beta \geq 0$  and  $\mu = f dV_M$  where  $f \in L^p(X, \mu)$  where  $M$  is a real smooth submanifold  $M$  of  $X$  which has codimension one and  $dV_M$  is the measure supported on  $M$  obtained by integrating against the Riemannian volume form on  $M$  induced by  $\omega_0$

*Proof.* Let  $\mu_0$  be a volume form and  $u \in \mathcal{E}^1(X, \omega)$  a solution to equation 3.2. *Step one:  $u$  is bounded (continuous).* Since  $u \in \mathcal{E}^1(X, \omega)$  the function  $u$  has no Lelong numbers ([52], Cor 1.8), i.e.  $\int e^{\beta u} \mu_0$  is integrable for all  $\beta$  (by Skoda's inequality, see for example [35]). In particular, by equation 3.2  $MA(u) \in L^p(X)$  for some  $p > 1$ . But then Kolodziej's theorem [58] says that  $u$  is bounded (and even Hölder continuous [59]).

*Step two: higher order regularity.* By the previous step  $u$  is a bounded weak solution to an equation of the form  $MA(u) = e^{\Phi(u)} \mu_0$  where  $\Phi(x)$  is a smooth function on  $\mathbb{R}$ . But then the theorem of Székelyhidi-Tosatti [79] says that  $u$  is smooth.  $\square$

When  $\beta = 0$  the first and second point is proved in [59] and [53], respectively. But then the case when  $\beta > 0$  also follows, since the factor  $f := e^{\beta u}$  is always bounded then (just using that  $u$  is usc).

**3.1. The case when  $\beta > 0$ .** We start by considering the general case when  $\beta > 0$  which is considerably simpler than the case when  $\beta < 0$ . This difference in behavior is a reflection of the fact that in the former case the functional  $F_\beta$  above is a *sum* of two convex functionals, while in the latter case it is a *difference* of two convex functionals.

**Theorem 3.2.** *Assume that  $\beta > 0$  and that the background measure  $\mu_0$  does not charge pluripolar sets. Then there is a unique solution  $u_{ME} \in \mathcal{E}_1(X)$  mod  $\mathbb{R}$  of the Monge-Ampère mean field equation 3.1. Moreover,  $u_{ME}$  is smooth if  $\mu_0$  is a volume form. In general,*

- $u_{ME}$  is the unique (mod  $\mathbb{R}$ ) maximizer of the functional  $\mathcal{G}_\beta$  on  $\mathcal{E}^1(X, \omega)$
- $\mu_{ME} (:= MA(u_{ME}))$  is the unique minimizer of the free energy functional  $F_\beta$  on  $\mathcal{M}_1(X)$

More generally, if  $\mu_j$  is a sequence such that

$$F_\beta(\mu_j) \rightarrow \inf_{E(\mu, \omega)} F_\beta$$

then  $\mu_j$  converges to  $\mu_{ME}$  in the weak topology of measures.

*Proof.* To simplify the notation we assume that  $\beta = 1$  and write  $\mathcal{G} := \mathcal{G}_1$  and  $\mathcal{L}^+(u) := \log \int e^u \mu_0$  so that  $\mathcal{G} = \mathcal{E} - \mathcal{L}^+$ .

*Existence of solution:*

The existence of a solution  $u_{ME} \in \mathcal{E}^1(X, \omega)$  follows from the variational approach to Monge-Ampère equations introduced in [10]. In that paper the case when  $\beta = 0$  was treated, as well as the case when  $\beta > 0$  and  $\mu_0$  is a volume form. But precisely the same arguments go through in the general case and realize  $u_{ME}$  as the unique (mod  $\mathbb{R}$ ) maximizer of the functional  $\mathcal{G}$  on  $\mathcal{E}^1(X, \omega)$ . Note that when  $n > 1$  this is highly non-trivial as one, a priori, has to optimize  $\mathcal{G}$  over a convex set with boundary. But as shown in [10] this problem can be circumvented by using the projection  $P$ . For completeness we recall the proof:

*Step one: existence of a maximizer of  $\mathcal{G}$*

We will denote by  $\mathcal{E}^1(X, \omega)_0$  the subspace of all  $u$  in  $\mathcal{E}^1(X, \omega)$  such that  $\sup_X u = 0$ . Since  $\mathcal{G}$  is invariant under the  $\mathbb{R}$ -action we may take a sequence in  $\mathcal{E}^1(X, \omega)_0$  such that

$$\mathcal{G}(u_j) \rightarrow \sup_{\mathcal{E}^1(X, \omega)} \mathcal{G} < \infty$$

Moreover, by the compactness of  $PSH(X)/\mathbb{R}$  (see section 2.2) we may assume that  $u_j \rightarrow u_\infty$  in  $L^1(X)$ . By Prop 2.4  $\mathcal{E}_\omega$  is usc and according to Lemma 1.14 in [11] so is  $\mathcal{L}^+$  since  $\mu_0$  does not charge pluripolar sets (see Lemma 3.6 below for a generalization). Hence  $u_\infty \in \mathcal{E}^1(X, \omega)$  and

$$\mathcal{G}(u_\infty) \geq \sup_{\mathcal{E}^1(X, \omega)} \mathcal{G} < \infty$$

and since  $u_\infty$  is a candidate for the sup equality must hold above.

*Step two: Any maximizer of  $\mathcal{G}$  on  $\mathcal{E}^1(X, \omega)$  satisfies equation 3.1*

Let  $u_*$  be a maximizer, fix  $v \in \mathcal{C}^\infty(X)$  and consider the following function on  $\mathbb{R}$

$$g(t) := \mathcal{E}(P(u_* + tv)) + \mathcal{L}^+(u_\infty + tv)$$

where  $(P(u_* + tv) \in \mathcal{E}^1(X, \omega)$ , since  $v$  is bounded. It has a global maximizer at  $t = 0$ . Indeed, this using that the projection  $P$  and  $-\mathcal{L}$  are increasing with respect to  $\leq$  gives

$$(\mathcal{E} \circ P) - \mathcal{L}(u) = (\mathcal{E} - \mathcal{L})(Pu) + \mathcal{L} \circ P - \mathcal{L} \leq (\mathcal{E} - \mathcal{L})(Pu)$$

Since by Theorem 2.6 (and an approximation argument [10]) and Prop 2.11  $g$  is differentiable it follows from the formulas for their derivatives that

$$\frac{dg(t)}{dt} \Big|_{t=0} = 0 = \left\langle MA(Pu_*) - e^{\beta u} / \int_X e^{\beta u} \mu_0, v \right\rangle = 0$$

and since, by definition,  $Pu_* = u_*$  and  $v$  was arbitrary this means that  $u_\infty$  solves equation 3.1.

*Regularity:*

By Prop 3.1 any weak solution as above is in fact smooth when  $\mu_0$  is a volume form. It should be pointed out that when  $\mu_0$  is a volume form the existence of a smooth solution, when  $\beta > 0$ , is a direct consequence of the Aubin-Yau estimates [2, 93], using the continuity method.

*Proof of the second point:  $MA(u_{ME})$  is the unique minimizer of  $F$  (and  $u_{ME}$  is the unique solution of equation 3.2)*

To prove this first observe that  $F(\mu)$  is strictly convex on  $\{F < \infty\}$ . Indeed,  $E(\mu)$  is clearly convex (as it can be realized as a Legendre-Fenchel transform) and it is well-known [33] that  $D(\mu)$  is strictly convex on  $\{D < \infty\}$ . Now fix  $\mu$  such that  $F(\mu) < \infty$  and consider the affine segment

$$\mu^t := \mu_{MF}(1-t) + t\mu =: \mu_{MF} + t\nu$$

Next let us prove that

$$\frac{dD(\mu^t)}{dt} \Big|_{t=0^+} = - \int_X \log(\mu_{MF}/\mu_0) \nu$$

But this follows from Prop 2.11, since the rhs above is finite. Indeed, by the Monge-Ampère mean field equation, it equals  $-\int_X u_{MF} \nu$  where, as shown above,  $\mathcal{E}(u_{MF}) > \infty$  and by assumption  $\nu$  is a difference of finite energy measures. Hence, the integral is finite according to Theorem 2.6. Now combining the formula for  $\frac{dD(\mu^t)}{dt} \Big|_{t=0^+}$  above with the convexity of  $D$  and the inequality for  $E$  in Prop 2.7 gives

$$F(\mu) \geq F(\mu_{ME}) + 0$$

for any  $\mu$  such that  $E(\mu)$  and  $D(\mu)$  are both finite. Moreover, the *strict* concavity of  $F$  discussed above shows that  $\mu_{ME} := MA(u_{ME})$  is the *unique* minimizer of  $F(\mu)$  on  $E_1(X, \omega)$ . The previous argument also gives that any solution  $u_1 \in \mathcal{E}^1(X, \omega)$  of equation 3.2 is such that  $MA(u_1)$  is a minimizer of  $F$ . As a consequence  $MA(u_0) = MA(u_1)$  for any two solutions and hence  $u_0 - u_1$  is constant according to Theorem 2.6. This finishes the proof of the second point.

To prove the final convergence recall that the functionals  $E$  and  $D$  arise as Legendre transforms and are in particular lower semi-continuous. As a consequence any weak limit point  $\mu_*$  of the sequence  $\mu_j$  is a minimizer of  $F(\mu)$ . But then it follows from the strict convexity used above (i.e. the uniqueness) that  $\mu_* = \mu_{ME}$ .  $\square$

**Theorem 3.3.** *Assume that  $\beta > 0$ . Then the following relations between the functionals  $F := F_\beta$  and  $\mathcal{G} := \mathcal{G}_\beta$  hold*

- For any  $u \in \mathcal{E}^1(X, \omega)$  we have

$$F(MA(u)) \geq \mathcal{G}(u)$$

and

$$F(e^{\beta u} \mu_0 / \int e^{\beta u} \mu_0) \geq \mathcal{G}(u)$$

Equality in any of the two inequalities above holds iff  $u$  is a solution of the Monge-Ampère mean field equation 3.1 (and hence equalities then hold in *both* inequalities above)

- Moreover,

$$\inf_{\mu \in E_1(X, \omega)} F(\mu) = \sup_{u \in \mathcal{E}^1(X, \omega)} \mathcal{G}(u) < \infty$$



*Proof.* We skip the proof of the first point as it is a trivial modification (obtained by changing a few signs) of the proof given below for the corresponding inequalities in Theorem 3.4. The first point then immediately gives

$$(3.3) \quad \inf_{\mu \in E_1(X, \omega)} F(\mu) \geq \sup_{u \in \mathcal{E}^1(X, \omega)} \mathcal{G}(u)$$

According to the previous theorem the infimum in the LHS above is attained precisely for  $\mu = MA(u)$  where  $u$  is the unique solution mod  $\mathbb{R}$  of the Monge-Ampère mean field equation 3.1 and similarly for the supremum in the RHS above. But then it follows from the equality case in the first point that equality in fact holds in 3.3.  $\square$

### 3.2. The case when $\beta < 0$ .

#### 3.2.1. Duality between $F$ and $\mathcal{G}$ .

**Theorem 3.4.** *The following relations between the functionals  $F := F_{-\gamma}$  and  $\mathcal{G} := \mathcal{G}_{-\gamma}$  hold*

- The suprema coincide

$$(3.4) \quad \sup_{\mu \in E_1(X, \omega)} F(\mu) = \sup_{u \in \mathcal{E}^1(X, \omega)} \mathcal{G}(u)$$

Moreover, if  $\mu_0$  is a volume form then the suprema above coincide with the suprema taken over the subspaces of volumes forms in  $E_1(X, \omega)$  and Kähler potentials (i.e  $\mathcal{H}(X, \omega)$ ) in  $\mathcal{E}^1(X, \omega)$ , respectively.

- Moreover

$$(3.5) \quad E(\mu_0) \leq \sup_{\mu \in E_1(X, \omega)} F(\mu)$$

In particular, if the sup above is *finite* then necessarily the measure  $\mu_0$  has finite energy:

$$E(\mu_0) < \infty$$

and moreover

$$(3.6) \quad \int e^{-\gamma u} \mu_0 < \infty$$

for any  $u \in \mathcal{E}^1(X, \omega)$  (if  $\mu_0$  is a volume form then 3.6 holds for *any*  $\gamma$  even if the sup above is infinite)

- More generally, the following inequalities hold for any  $u \in \mathcal{E}^1(X, \omega)$

$$(3.7) \quad F(MA(u)) \leq \mathcal{G}(u)$$

and

$$(3.8) \quad F(e^{-\gamma u} \mu_0 / \int e^{-\gamma u} \mu_0) \geq \mathcal{G}(u)$$

Equality in any of the two inequalities above holds iff  $u$  is a solution of the Monge-Ampère mean field equation 3.1 with  $\beta = -\gamma$  (and hence equalities then hold in *both* inequalities above). Moreover, when  $n = 1$  the inequality 3.8 holds for any continuous function  $u$  on  $X$ .

*Proof. (of the first point):*

First recall the Legendre transform relations  $E(\mu) = (\mathcal{E} \circ P)^*$  and  $\frac{1}{\gamma} D(\mu) = \mathcal{L}_{\gamma}^-(\mu)^*$  (see section 2.6). Let us first prove

$$(3.9) \quad \sup_{\mu \in E_1(X, \omega)} F(\mu) \geq \sup_{u \in \mathcal{E}^1(X, \omega)} \mathcal{G}(u)$$

For the sake of notational simplicity we assume that  $\gamma = 1$  and simply write

$$\mathcal{L}(u) := \mathcal{L}_{-\gamma}^-(u) := -\log \int_X e^{-u} \mu_0$$

defining a concave functional on  $C^0(X)$ . First note that it follows immediately from the definition of the Legendre transforms that,

$$(\mathcal{E} \circ P) - \mathcal{L}(u) \geq c \text{ (on } C^0(X)) \Rightarrow (\mathcal{E} \circ P)^*(\mu) - \mathcal{L}^*(\mu) \geq c$$

and hence

$$\sup_{\mu \in E_1(X, \omega)} F(\mu) \geq \sup_{u \in C_0(X)} ((\mathcal{E} \circ P) - \mathcal{L})(u)$$

Next, observe that

$$\sup_{u \in C^0(X)} ((\mathcal{E} \circ P) - \mathcal{L})(u) = \sup_{u \in \mathcal{H}(X, \omega)} (\mathcal{E} - \mathcal{L})(u)$$

where the sup in the rhs may also be taken over  $\mathcal{E}^1(X, \omega)$ . Indeed, first using that the projection  $P$  and  $\mathcal{L}$  are increasing with respect to  $\leq$  gives

$$(\mathcal{E} \circ P) - \mathcal{L}(u) = (\mathcal{E} - \mathcal{L})(Pu) + \mathcal{L} \circ P - \mathcal{L} \leq (\mathcal{E} - \mathcal{L})(Pu)$$

Hence comparing the value at  $u$  in the lhs below with the value at  $Pu$  in the rhs below gives

$$\sup_{u \in C^0(X)} ((\mathcal{E} \circ P) - \mathcal{L})(u) = \sup_{u \in C^0(X) \cap \mathcal{E}^1(X, \omega)} (\mathcal{E} - \mathcal{L})(u)$$

Finally, by Demailly's approximation theorem [34] any  $u \in \mathcal{E}(X, \omega)$  can be written as a decreasing limit of elements in  $\mathcal{H}(X, \omega)$ . Hence, by the continuity of  $\mathcal{E}$  under such limits and Lebesgue's monotone convergence theorem the restriction to  $C^0(X)$  in the rhs above may be removed, finishing the proof of the claim 3.9.

The reversed inequality in 3.9 is proved by interchanging the roles of  $E(= (\mathcal{E} \circ P)^*)$  and  $(\mathcal{E} \circ P)$  and the roles of  $\mathcal{L}^*$  and  $\mathcal{L}$  and using the duality relations in Proposition 2.10 and section 2.6. This gives, just as above,

$$\sup_{\mu \in E_1(X, \omega)} F(\mu) \leq \sup_{u \in C^0(X)} ((\mathcal{E} \circ P) - \mathcal{L})(u) = \sup_{u \in \mathcal{H}(X, \omega)} (\mathcal{E} - \mathcal{L})(u)$$

which finishes the proof of the inequality in the first point of the theorem. The fact that the sup over  $E_1(X, \omega)$  may be taken over the subspace of volume forms will be given in the proof of the third point.

To prove the inequalities 3.5 and 3.6 note that assuming that the sup in question equals  $C < \infty$  gives

$$\frac{1}{\gamma} \log \int e^{-\gamma u} \mu_0 \leq \mathcal{E}_\omega(u) + C$$

proving 3.6. Moreover, by Jensen's inequality,

$$\mathcal{E}_\omega(u) - \int_X u \mu_0 \leq \mathcal{G}(u)$$

Taking the sup over all  $u \in \mathcal{E}^1(X, \omega)$  it follows from the definition of  $E(\mu_0)$  and the equality 3.4 that the inequality 3.5 indeed holds.

*Proof of the second point:*

Let us first prove that if  $u_\mu \in \mathcal{E}^1(X, \omega)$ , then

$$(3.10) \quad F(MA(u_\mu)) \leq \mathcal{G}(u_\mu)$$

with equality iff  $u$  solves equation 3.1. To this end write  $\mu := MA(u_\mu) = d\mathcal{E}|_{u_\mu}$ . Then, by definition,

$$F(\mu) = \mathcal{E}(u_\mu) - \langle u_\mu, \mu \rangle - \sup_{u \in C_0(X)} (\mathcal{L}(u) - \langle u, \mu \rangle) \leq \mathcal{E}(u_\mu) - \langle u_\mu, \mu \rangle - (\mathcal{L}(u_j) - \langle u_j, \mu \rangle)$$

for any  $u_j \in C_0(X)$ . In particular, taking continuous functions  $u_j$  decreasing to  $u_\mu$  and letting  $j \rightarrow \infty$  and using the monotone convergence theorem proves the inequality 3.10. Moreover, equality above clearly holds iff  $u_\mu$  realizes the sup defining  $\mathcal{L}^*(\mu)$ . By Lemma 2.12 this happens, since we assume that  $\int e^{-u_\mu} \mu_0$  is finite, iff

$$\mu = e^{-u_\mu} / \int e^{-u_\mu} \mu_0$$

which finishes the proof of the equality case in 3.10.

Next, to prove the inequality 3.8 first observe that, as explained above, setting  $\mu' := e^{-u'} / \int e^{-u'} \mu_0$  with  $u' \in \mathcal{E}^1(X, \omega)$  gives

$$F(\mu') = \sup_{u \in \mathcal{E}^1(X, \omega)} (\mathcal{E}(u) - \langle u, \mu' \rangle) - (\mathcal{L}(u') - \langle u', \mu' \rangle) \geq \mathcal{E}(u') - \mathcal{L}(u') = \mathcal{G}(u')$$

since  $u'$  is a candidate for the sup. Moreover, by Theorem 2.6 equality holds iff  $MA(u') = \mu'$  which means that  $u'$  is a solution of the Monge-Ampère mean field equation 3.1. As for the case when  $n = 1$  we take  $u'$  continuous, but without assuming  $\omega_{u'} \geq 0$ . We can then repeat the same argument as above but taking the sup above over  $C^0(X)$  instead of  $\mathcal{E}^1(X, \omega)$  (see remark 2.8).

Finally, let us prove the last statement in the proof of the first point. As explained in the proof of the first point we have

$$\sup_{E_1(X, \omega)} F(\mu) \leq \sup_{u \in \mathcal{H}(X, \omega)} \mathcal{G}(u)$$

But as shown above we have, if  $u \in \mathcal{H}(X, \omega)$  and  $\mu_0$  is a volume form that

$$\mathcal{G}(u) \leq F(e^{-\gamma u} / \int e^{-\gamma u} \mu_0) \leq \sup_{V(X, \omega)} F(\mu)$$

where  $V(X, \omega)$  denotes the subspace of volume forms.  $\square$

**3.2.2. Intermezzo: properness vs coercivity.** Before continuing we will briefly discuss some relations between properness and coercivity of the functionals  $\beta F_\beta$  and  $\beta G_\beta$  that will not be used elsewhere. It follows immediately from inequality 3.7 above that if  $\beta G_\beta$  is proper (wrt energy) then so is  $\beta F_\beta$ . It would be interesting to know if the converse is true? In the Kähler-Einstein setting this was indeed shown by Tian, see [86] (Thm 7.13). The proof is indirect and uses the continuity method to first establish the existence of a Kähler-Einstein metric  $\omega_{KE}$ . Using the existence of  $\omega_{KE}$ , reversing the continuity method and also smoothing by the Kähler-Ricci flow then gives the properness of  $\beta G_\beta$  (in this case  $\beta = -1$ ). As conjectured by Tian and subsequently established in [70] the previous argument can be refined to give that  $\beta G_\beta$  is even coercive. All in all this in particular shows that  $\beta F_\beta$  is coercive iff  $\beta G_\beta$  is. As next observed this latter property can be obtained as a corollary of Theorem 3.4 in the setting of a *general* measure  $\mu_0$ :

**Corollary 3.5.** *Let  $\mu_0$  be a measure on  $X$  and  $\beta (= -\gamma)$  a negative number. Then the corresponding functional  $\beta F_\beta$  is coercive iff  $\beta G_\beta$  is coercive.*

*Proof.* Assume that  $\beta F_\beta$  is coercive or equivalently that  $F_{\beta(1+\delta)}$  is bounded from above for some  $\delta > 0$ . Then it follows from Theorem 3.4 that  $G_{\beta(1+\delta)}$  is also bounded from above, i.e. for any  $\omega$ -psh function  $v$  we have

$$\frac{1}{\gamma(1+\delta)} \log \int e^{-\gamma(1+\delta)v} \mu_0 \leq -\mathcal{E}_\omega(v) + C$$

To prove coercivity for  $\beta G_\beta(u)$  we let  $u$  be an arbitrary  $\omega$ -psh function. By scale invariance it will be enough to consider the case  $\int u \omega^n = 0$ , so that  $-\mathcal{E}_\omega(u) = J_\omega(u)$ . Then  $v := u/(1+\delta)$  is also  $\omega$ -psh function (since  $\delta > 0$ ) such that  $\int v \omega^n = 0$  and hence applying the previous inequality to  $v$  gives

$$\frac{1}{\gamma} \log \int e^{-\gamma u} \mu_0 - C \leq (1+\delta) J_\omega(u/(1+\delta)) \leq (1+\delta)^{-1/n} J_\omega(u),$$

where the last inequality follows from  $J_{\omega_0}(tu) \leq t^{1+1/n} J_{\omega_0}(u)$  if  $0 < t < 1$  (see remark 2 in [39]). Since,  $(1+\delta)^{-1/n} < 1$  this shows that  $-G_{-\gamma}$  is also coercive. The reversed implication follows immediately from Theorem 3.4.  $\square$

**3.2.3. A continuity lemma.** We will next prove a useful continuity result, using a very minor modification of the proof of Lemma 1.14 in [11]. In the case when  $\mu_0$  is a volume form the lemma follows from results of Demailly-Kollar [35] (see Theorem 6.1 in the appendix).

**Lemma 3.6.** *Assume that  $\psi_j \rightarrow \psi$  in  $PSH(X, \omega)$  and that there is a positive number  $\delta$  such that*

$$\int_X e^{-(\gamma+\delta)\psi_j} \mu_0 \leq C$$

*where the measure  $\mu_0$  does not charge pluripolar sets. Then*

$$(3.11) \quad \int_X e^{-\gamma\psi_j} \mu_0 \rightarrow \int_X e^{-\gamma\psi} \mu_0$$

*for any real number  $\gamma$ .*

*Proof.* Let  $u_j := e^{-\gamma\psi_j} \mu_0$ . By assumption there is a constant  $C$  and  $p > 1$  such that

$$\|u_j\|_{L^p(\mu_0)} \leq C$$

Hence, it follows from general functional analysis (using that the unit ball in  $L^p(\mu)$  is weakly compact and the Hahn-Banach separation theorem; compare the proof of Lemma 1.14 in [11]) that if

$$\|u_j\|_{L^1(\mu_0)} \rightarrow l \in \mathbb{R}$$

then, after perhaps replacing  $u_j$  with a subsequence, there are numbers  $c_j$  and  $v \in L^p(\mu_0)$  such that

$$\sum_j c_j = 1, \quad \sum_j c_j u_j \rightarrow v \text{ in } L^p(\mu_0)$$

By the previous convergence this forces

$$\int v \mu_0 = l$$

Moreover, after perhaps again replacing  $u_j$  with a subsequence, it follows from general integration theory that we may assume that

$$(3.12) \quad \sum_j c_j u_j \rightarrow v \text{ a.e. wrt } \mu_0$$

Now, since  $\mu$  does not charge pluripolar set Hartog's lemma [51] gives that  $\limsup \psi_j = \psi$  a.e. wrt  $\mu_0$ . But this fact combined with 3.12 then forces  $v = e^{-\gamma\psi}$  a.e. wrt  $\mu_0$ . All in all this means that

$$\|u_j\|_{L^1(\mu_0)} \rightarrow l = \int e^{-\gamma\psi} \mu_0$$

Since the limit is independent of the choice of subsequences above this finishes the proof of the lemma.  $\square$

**3.2.4. Existence of optimizers for the free energy and converge towards optimizers.** Next, we will prove one of the main results of the present paper showing that coercivity of the free energy functional  $F$  is sufficient for the existence of a maximizer.

**Theorem 3.7.** *Let  $\beta = -\gamma < 0$ . Suppose that the functional  $-F_{-\gamma}$  is coercive (wrt energy) or equivalently that  $F_{-\gamma-\delta}$  is bounded for some  $\delta > 0$ . Then  $F_{-\gamma}$  admits a finite energy maximizer  $\mu_\beta$ . Moreover, the potential of any maximizer solves the Monge-Ampère mean field equation 3.1. More generally, if  $\mu_j$  is a sequence such that*

$$F_{-\gamma}(\mu_j) \rightarrow \sup_{E_1(X, \omega)} F_{-\gamma} < \infty$$

*Then, perhaps after passing to a subsequence  $\mu_j$  converges weakly to a maximizer  $\mu_\beta$ . If  $\mu_0$  is a volume form then the maximizer is smooth.*

*Proof.* Let  $\mu_j$  be a maximizing sequence for  $F_{-\gamma}$ , as in the assumptions above. By the boundedness assumption of  $F_{-(\gamma+\delta)}$  and by the maximizing property it follows immediately that  $E(\mu_j) \leq C$ . Writing  $\mu_j = MA(u_j)$  this means according to Lemma 2.13, that  $(I - J)(u_j)$  and hence  $J(u_j)$  are uniformly bounded:

$$J(u_j) \leq C_\delta$$

Combining this latter bound with the fact that  $G_{\gamma+\delta}$  is also bounded from above (by the first point in Theorem 3.4) gives that

$$(3.13) \quad \int_X e^{-(\gamma+\delta)u_j} \mu_0 \leq C_\delta$$

and hence after adjusting by constants to get  $\sup u_j = 0$  and passing to a subsequence to make sure that  $u_j \rightarrow u$  in  $L^1$ , the convergence 3.11 in Lemma 3.6 gives, also using that  $\mathcal{E}$  is usc (Prop 2.4)

$$\infty > G_{-\gamma}(u) \geq \limsup G_{-\gamma}(u_j)$$

Combining this with the first and second point in Theorem 3.4 gives

$$\sup_{\mathcal{E}^1(X, \omega)} G_{-\gamma} = \sup_{E_1(X, \omega)} F_{-\gamma} = \limsup F_{-\gamma}(\mu_j) \leq \limsup \mathcal{G}_{-\gamma}(u_j) \leq G_{-\gamma}(u) < \infty$$

and hence  $u$  is a maximizer of  $G_{-\gamma}$  on  $\mathcal{E}^1(X, \omega)$ . But then it follows precisely as in the proof of Theorem 3.2 above, using the projection operator  $P$ , that  $u$  is a solution of equation 3.1.  $\square$

**3.3. The proof of Theorem 1.2 and a refined version.** Apart from the last statement in the theorem concerning properness the proof follows immediately from combining the theorems in this section established above. Finally, in the general case when  $F_{-\gamma}$  is only assumed proper the previous still applies as long as  $\mu_0$  satisfies the following qualitative Moser-Trudinger type inequality:

$$(3.14) \quad J(u) \leq C \implies \int e^{-(\gamma+\delta)(u - \sup u)} \mu_0 \leq C_{\gamma, \delta}$$

This inequality does hold in the case when  $\mu_0 = f dV$  with  $f \in L^p(X, dV)$  for  $p > 1$  as follows immediately from Hölder's inequality and the following stronger property of any volume form  $dV$  :

$$(3.15) \quad J(u) \leq C \implies \int I_t(u) := \int e^{-t(u \cdot \sup u)} dV \leq C_t.$$

for any  $t > 0$  obtained in the proof of Lemma 6.4 in [10], using Zeriahi's uniform variant of Skoda's theorem [94] (see the proof of Lemma 6.4 in [10]). More generally, the previous arguments shows the that the following refined version of the last part of Theorem 1.2 holds:

**Theorem 3.8.** *Assume that  $\mu_0$  satisfies the qualitative Moser-Trudinger type inequality 3.14 and let  $u_j$  be a sequence in  $\mathcal{E}^1(X, \omega)$  such that  $J_\omega(u_j) \leq C$  (or equivalently,  $\mathcal{E}_\omega(u - \sup u_j) \geq -C'$ ). Then  $F_{-\gamma}(u_j)$  is uniformly bounded from above. If furthermore  $u_j$  is a maximizing sequence for  $F_{-\gamma}$  then  $u_j - \sup u_j$  converges (after perhaps passing to a subsequence) to a maximizer for  $F_{-\gamma} \circ MA$ .*

*Remark 3.9.* It may be worth pointing out that the convergence  $I_t(u_j) \rightarrow I(u)$  used in the proof above can also be deduced from the results of Demailly-Kollar. Indeed, since  $J(u_j) \leq C$  and we may assume that  $u_j \rightarrow u$  in  $L^1(X)$ , the fact that  $\mathcal{E}$  is usc (and hence  $J$  is lsc) gives  $J(u) \leq C < \infty$ . But then  $u$  has no Lelong numbers (as follows from Cor 1.4 in [52]) and hence  $I_t(u) < \infty$  for all  $t$  (compare the proof of prop 3.1). But then it follows from Theorem 6.1 that  $I_t(u) \rightarrow I(u)$  (compare the proof of Cor 6.2).

**3.4. Alpha invariants.** We define the generalized  $\alpha$ -invariant of a pair  $([\omega], \mu_0)$  by

$$\alpha([\omega], \mu_0) := \sup \left\{ \alpha : \exists C_\alpha : \int_X e^{-\alpha(u - \sup_X u)} \mu_0 \leq C_\alpha, \forall u \in PSH(X, \omega) \right\}$$

**Example 3.10.** If  $(X, \omega)$  is a Riemann surface with  $\int_X \omega = 1$  then  $\alpha([\omega]) = 1$ . Indeed, if we denote by  $g_{x_0}$  the corresponding *Green function with a pole at  $x_0$*  defined by

$$dd^c g_{x_0} = \delta_{x_0} - \omega, \quad \int g_{x_0} \omega = 0$$

where  $\delta_{x_0}$  is the Dirac mass at the point  $x_0$  then the integral  $\int_X e^{-\alpha(u - \sup_X u)} \mu_0$  for  $u = g_{x_0}$  is finite for  $\alpha < 1$  and infinite for  $\alpha = 1$  (as follows from the standard fact that  $g_{x_0} - \log d^2(x, x_0) \in C^0(X)$  in terms of the distance function wrt the metric  $\omega$ ). Decomposing a general element  $u \in PSH(X, \omega)$  as  $u(x) = \int u(y) g_y(x) \omega(y)$  and using Jensen's inequality then proves the claim. Similarly, if there are positive constants  $C$  and  $d$  such that the measure  $\mu_0$  satisfies

$$\mu(B_r) \leq C r^d,$$

for  $r$  sufficiently small, for every geodesic ball of radius  $r$ , then

$$\alpha \geq 2d.$$

**3.4.1. Proof of Theorem 1.3.** By Theorems 3.7 and 3.4 it will be enough to prove that  $F_{-\gamma}$  is coercive under the assumptions of the theorem. To this end first note that by assumption we have that

$$\mathcal{L}_t^-(u) > -C$$

for any fixed  $t$  with  $t < \alpha$ . Writing  $\mu = MA(u_\mu)$  for the potential  $u_\mu$  such that  $\sup u_\mu = 0$  gives

$$\frac{1}{t}D(\mu) = \sup_u \mathcal{L}_t^-(u) - \langle u, \mu \rangle \geq \mathcal{L}_t^-(u_\mu) - \langle u_\mu, \mu \rangle \geq -\langle u_\mu, \mu \rangle - C,$$

i.e.

$$D(\mu) \geq -t \langle u_\mu, \mu \rangle - C$$

This means that

$$F_{-\gamma}(\mu) \leq E(\mu) + \frac{t}{\gamma} \langle u_\mu, \mu \rangle + C$$

Combining the previous inequality with the inequality 2.20 hence gives

$$F_{-\gamma}(\mu) \leq E(\mu)(1 - \frac{t}{\gamma}(\frac{n+1}{n})) + C,$$

showing that  $F_{-\gamma}$  is proper and even coercive (wrt energy) as long as

$$\gamma < \alpha(\frac{n+1}{n})$$

and  $t$  is chosen sufficiently close to  $\alpha$ . Hence the theorem follows from Theorem 1.2.

**3.5. Proof of Corollary 1.4.** Let us first prove that when  $n = 1$  the bound on  $\mathcal{G}_{-\gamma}(v)$  in fact holds for all smooth functions  $v$  on  $X$ . This can be seen in two ways. First, it follows precisely as in the proof Cor 3 in [7] from using the following inequality for  $v \in \mathcal{C}^\infty(X)$  proved there:

$$\mathcal{E}_\omega(P_\omega v) \geq \mathcal{E}_\omega(v)$$

(which is a rather direct consequence of the orthogonality relation 2.13 when  $n = 1$ ). Combining the previous inequality with the fact that  $\mathcal{L}_\gamma^-(u)$  is increasing in  $u$  immediately gives

$$\sup_{v \in \mathcal{C}^\infty(X)} \mathcal{G}_{-\gamma}(v) \leq \sup_{v \in \mathcal{C}^\infty(X)} \mathcal{G}_{-\gamma}(P_\omega v) \leq \sup_{\mathcal{H}(X, \omega)} \mathcal{G}_{-\gamma}(v)$$

which is bounded by Theorem 1.3.

Alternatively, for  $v$  continuous we let  $\mu := e^{-\gamma v} / \int e^{-\gamma \cdot v} \mu_0$ . Then, by the last point in Theorem 3.4

$$\mathcal{G}_{-\gamma}(v) \leq F_\gamma(e^{-\gamma v} / \int e^{-\gamma v} \mu_0) \leq C$$

using Theorem 1.3 in the last step (in the Kähler-Einstein setting on  $S^2$  a similar argument was used by Rubinstein [71]).

Finally, since if  $\int u \omega = 0$  and  $\int \omega = 1$  we have

$$\mathcal{E}_\omega(u) = -\frac{1}{2} \int_X du \wedge d^c u$$

and hence  $\mathcal{E}_\omega(cu) = c^2 \mathcal{E}_\omega(u)$ . All in all this means that we obtain the inequality we wanted to prove from  $\mathcal{G}_{-\gamma}(\frac{1}{\gamma}u) \leq C$

*Remark 3.11.* Even though formulated for Riemann surfaces without boundary Corollary 1.4 also contains the analogous statement on any compact Riemann surface  $Y$  with smooth boundary  $\partial Y$  if one demands, as usual, that  $u = 0$  on  $\partial Y$ . Indeed, if  $Y$  is a domain in the compact closed Riemann surface  $X$  and  $u \in C^0(Y)$  with  $y = 0$ , or more generally  $u$  is in the Sobolev space  $H_0^1(Y)$  (i.e. the closure in the Dirichlet norm of the space  $C_0^\infty(Y)$  of all smooth and compactly supported functions on the interior of  $Y$ ) it is, by standard continuity arguments, enough to

prove the inequality for  $u \in C_0^\infty(Y)$ . Extending by zero gives  $u \in \mathcal{C}^\infty(X)$  and then the inequality then follows immediately from Corollary 1.4 when  $\omega$  is taken as a measure supported on  $\partial Y$  in  $X$ .

**3.6. The limit  $\beta \rightarrow \infty$  : envelopes and free boundaries.** In this section we will take the fixed form  $\omega$  on  $X$  to be any smooth and closed  $(1,1)$ -form defining a Kähler *class* in  $H^{1,1}(X, \mathbb{R})$  (but not necessarily a Kähler *form*). Consider the following free boundary value problem for a function  $u$  on  $X$  :

$$(3.16) \quad \begin{aligned} (\omega + dd^c u)^n &= 0 && \text{on } \{u < 0\} \\ u &\leq 0 && \text{on } X \\ \omega_u &\geq 0 && \text{on } X \end{aligned}$$

It follows immediately from the domination principle for the Monge-Ampère operator (see Cor 2.5 in [14]) that the solution is unique and can be represented as an upper envelope:

$$(3.17) \quad P_\omega 0 = \sup_{v \in PSH(X, \omega)} \{v(x) : v \leq 0 \text{ on } X\}$$

**Theorem 3.12.** *Given a volume form  $\mu_0$  on  $X$  and  $\beta > 0$  let  $v_\beta$  the unique solution of the non-normalized Monge-Ampère mean field equation 3.1 and  $u_\beta$  the unique solution of equation 3.2 normalized so that  $\sup_X u_\beta = 0$ . Then both  $u_\beta$  and  $v_\beta$  converge in  $L^1(X)$  to a the solution of the free boundary value problem 3.16, which in turn coincides with the envelope  $P_\omega 0$  above.*

*Proof.* Let  $\mathcal{L}_\beta^+(u) := \frac{1}{\beta} \log \int_X e^{\beta u_\beta} \mu_0$ .

*Step 1: Convergence of  $u_\beta$*

Since  $\mu := MA(P_\omega 0)$  is a candidate for the sup defining the Legendre transform of  $D_{\mu_0}$  we get (see section 2.6 or use directly Jensen's inequality)

$$-\int_X u_\beta MA(P_\omega 0) + \frac{1}{\beta} D_{\mu_0}(MA(P_\omega 0)) \geq -\frac{1}{\beta} \mathcal{L}_\beta^+(u_\beta)$$

Hence defining the constant  $D := D_{\mu_0}(MA(P_\omega 0))$  gives

$$(3.18) \quad \begin{aligned} \mathcal{E}_\omega(u_\beta) - \int_X u_\beta MA(P_\omega 0) + \frac{D}{\beta} &\geq \mathcal{E}_\omega(u_\beta) - \frac{1}{\beta} \mathcal{L}_\beta^+(u_\beta) \geq \\ &\geq \mathcal{E}_\omega(P_\omega 0) - \mathcal{L}_\beta^+(P_\omega 0) \end{aligned}$$

using, in the last inequality that, by Theorem 3.2b  $u_\beta$  maximizes the functional  $\mathcal{G}_\beta$ . Since

$$\mathcal{L}_\beta^+(P_\omega 0) \rightarrow \sup_X P_\omega 0 = 0$$

(the last equality above follows for example from the orthogonality relation 2.13) this means that

$$(3.19) \quad \liminf_{\beta \rightarrow \infty} \mathcal{E}_\omega(u_\beta) - \int_X (u_\beta MA(P_\omega 0)) \geq \mathcal{E}_\omega(P_\omega 0) - \int_X (P_\omega 0) MA(P_\omega 0)$$

also using the orthogonality relation 2.13 saying that the second term in the rhs vanishes. But by the last statement in Theorem 2.6 it then follows that  $u_\beta \rightarrow P_\omega 0$  in  $L^1(X)$  and that 3.19 is actually an equality when  $\liminf$  is replaced by  $\lim$ .

*Step two: Convergence of  $v_\beta$*

By the asymptotic equality referred to above combined with the fact that  $u_\beta \rightarrow P_\omega 0$  and the orthogonality relation we get the following “convergence in energy”:

$$\mathcal{E}_\omega(u_\beta) \rightarrow \mathcal{E}_\omega(P_\omega 0)$$



Hence, using the orthogonality relation 2.13 again the inequalities 3.18 force

$$-\frac{1}{\beta}\mathcal{L}_\beta^+(u_\beta) \rightarrow 0$$

i.e.  $v_\beta := u_\beta - \frac{1}{\beta}\mathcal{L}_\beta^+(u_\beta)$  has the same limit as  $u_\beta$  and satisfies the equation 3.2.  $\square$

As shown in [12] the envelope  $P_\omega 0$  has a Laplacian which locally bounded it hence seems natural to ask if the convergence above holds in the Hölder space  $C^{1,\alpha}(X)$  for any  $\alpha < 1$ ?

#### 4. THE (TWISTED) KÄHLER-EINSTEIN SETTING

In this section the measure  $\mu_0$  will be taken to be a volume form and we will then reformulate equation 1.1 as a twisted Kähler-Einstein equation. First recall that the *Ricci curvature* of a Kähler metric is defined, in local holomorphic coordinates, by

$$\text{Ric}\omega := dd^c(-\log(\frac{\omega^n}{(i\sum_j dz_j \wedge d\bar{z}_j)^n})) (= -dd^c \log(\det \omega_{ij}))$$

representing the anti-canonical class  $-c_1(K_X)$ . If  $\theta$  is a given closed  $(1,1)$ -form on  $X$  the *twisted Kähler-Einstein equation* for a Kähler metric  $\omega$  is defined by

$$(4.1) \quad \text{Ric}\omega - \theta = -\beta\omega \quad (\gamma := -\beta \in \mathbb{R})$$

where, compared with the previous notation and the lhs is called the *twisted Ricci curvature* of  $\omega$ . It hence implies the following cohomological relation in  $H_{dd^c}^2(X, \mathbb{R})$ :

$$(4.2) \quad [\omega] = \beta(c_1(K_X) + [\theta])$$

and hence it forces  $\beta(c_1(K_X) + [\theta])$  to be a Kähler class which we will henceforth assume. Fix a Kähler form  $\omega = \omega_0$  in  $\beta(c_1(K_X) + [\theta])$ . We define its *twisted Ricci potential*  $h = h_{\omega,\theta}$  by the following equation

$$(4.3) \quad \text{Ric}\omega - \theta = -\beta(\omega + dd^c h_{\omega,\theta})$$

(which is solvable by the cohomological relation above) where the normalization constant is fixed by imposing  $\int_X e^{-h_{\omega,\theta}} \omega^n = 1$ . Subtracting the previous equation from equation 4.1 (with  $\omega = \omega_u$ ) hence gives the following equation for  $u$ :

$$dd^c(-\log(\frac{\omega_u^n}{\omega^n})) = dd^c(-\beta u + \beta h_{\omega,\theta})$$

which in turn is equivalent to the equation

$$(4.4) \quad (\omega + dd^c u)^n = e^{-\beta h_{\omega,\theta}} e^{\beta u} \omega^n,$$

i.e. the Monge-Ampère mean field equation 3.2 with

$$(4.5) \quad \mu_0 = e^{-\beta h_{\omega,\theta}} \frac{\omega^n}{Vn!}$$

and  $[\omega] = \beta(c_1(K_X) + [\theta])$ . We will call this particular choice of a triple  $(\beta, \omega, \mu_0)$  for the *twisted Kähler-Einstein setting*. In fact, the previous argument shows that the equation 3.2 is equivalent to the twisted Kähler-Einstein equation when  $\mu_0$  is a volume form, as follows by first defining  $h_{\omega,\theta}$  by the relation 4.5 and then  $\theta$  by the relation 4.3.

**4.1. The twisted Mabuchi functional as the free energy.** Next, we define, for a fixed  $\beta$ , the following functional in the twisted Kähler-Einstein setting the twisted Mabuchi functional by

$$\mathcal{K}_\theta(u) := \beta F_\beta(MA(u))$$

**Proposition 4.1.** *The functional  $\mathcal{K}_\theta(u_t)$  coincides with the twisted Mabuchi functional introduced in [75], i.e.*

$$d\mathcal{K}_{|\theta}|_u = (\beta\omega_u - (\text{Ric } \omega_u - \theta)) \wedge \frac{\omega_u^{n-1}}{(n-1)!}$$

and  $\mathcal{K}_\theta$  can hence be decomposed as  $\mathcal{K}_\theta = \mathcal{K} + \mathcal{J}_\theta$  where

$$d\mathcal{K}|_u = (\beta\omega_u - \text{Ric } \omega_u) \wedge \frac{\omega_u^{n-1}}{(n-1)!}, \quad d\mathcal{J}_\theta|_u = \theta \wedge \frac{\omega_u^{n-1}}{(n-1)!}$$

*Proof.* Combining Proposition 2.7 and 2.11 gives

$$\frac{d\mathcal{K}(u_t)}{dt} = \int (-\beta u_{MA(u_t)} + \log(\frac{MA(u_t)}{\mu_0})) \frac{dMA(u_t)}{dt}$$

Now  $\frac{dMA(u_t)}{dt} = dd^c(\frac{du_t}{dt}) \wedge \omega_{u_t}^{n-1}/(n-1)!$  and hence integration by parts give

$$\begin{aligned} \frac{d\mathcal{K}(u_t)}{dt} &= \int \frac{du_t}{dt} dd^c(-\beta u_{MA(u_t)} + \log(\frac{MA(u_t)}{\mu_0})) \wedge \omega_{u_t}^{n-1}/(n-1)! = \\ &= \int \frac{du_t}{dt} (-\beta\omega_u + (\beta\omega + dd^c \log(\frac{MA(u_t)}{\mu_0}))) \wedge \omega_{u_t}^{n-1}/(n-1)! \end{aligned}$$

using that, by definition,  $\omega_{u_{MA(u_t)}} = \omega_{u_t}$ . Finally, since the second term in the sum above may be written as  $(\beta\omega)_{\log(\frac{MA(u_t)}{\mu_0})} = -\text{Ric}\omega_t + \theta$  when  $\mu_0 = e^{\beta h_{\omega, \theta}} \omega^n / Vn!$  this proves the formula above for  $d\mathcal{K}_\theta$ .  $\square$

The functional  $\mathcal{K}$  appearing in the previous proposition is the usual (untwisted) Mabuchi functional and the functional  $\mathcal{J}_\theta$  was studied, for example, in [42] (section 4.3) and in [27]. As shown by Mabuchi [65] and Donaldson  $\mathcal{K}$  is convex along geodesics in  $\mathcal{H}_\omega(X)$  (defined in terms of the *Mabuchi-Semmes-Donaldson Riemannian metric*  $g$  on  $\mathcal{H}(X, \omega)$ ; see section 4.2 below). Using this latter convexity we also deduce the following

**Proposition 4.2.** *If  $\theta \geq 0$  is a positive current then the functional  $\mathcal{K}_\theta(u_t)$  is convex along geodesics  $u_t$  in  $\mathcal{H}_\omega(X)$  and strictly convex if  $\theta$  is a Kähler current, i.e.  $\theta > \epsilon\omega_0$ . Moreover, if  $\theta$  is a positive multiple of the current of integration  $\delta_D$  along an irreducible smooth divisor  $D$ , then  $d^2\mathcal{K}_\theta(u_t)/d^2t = 0$  at a given  $t$  iff  $\bar{\partial}V_t = 0$  and  $V_t$  is tangential to  $D$ . In particular,  $d^2\mathcal{K}_\theta(u_t)$  is geodesically strictly convex if  $X$  admits no non-trivial holomorphic vector fields which are tangent to  $D$ .*

*Proof.* The first part was already observed by Stoppa [77] and hence we consider the case when  $\theta = c\delta_D$  (and it will be clear that we may assume that  $c = 1$ ). Let us first recall the following formula for a geodesic  $u_t$  :

$$(4.6) \quad \partial_t^2 u_t - |\bar{\partial}(\partial_{\bar{t}} u)|_{\omega_{u_t}}^2 (= \partial_t^2 u_t - |V_t|_{\omega_{u_t}}^2) = 0,$$

where  $V_t$  is the  $(1, 0)$ -vector field which is dual to the  $(0, 1)$ -form  $\bar{\partial}(\partial_{\bar{t}} u)$  under  $\omega_{u_t}$ . We also recall the following formula [65, 41] of the usual Mabuchi functional along a geodesic:

$$\frac{\partial^2 \mathcal{K}(u_t)}{\partial^2 t} = \int_X |\bar{\partial} V|^2_{\omega_{u_t}} \frac{\omega_{u_t}^n}{n!} (\geq 0)$$

Next, a direct calculation gives

$$\frac{\partial^2 \mathcal{J}_\theta(u_t)}{\partial^2 t} = \int_D (\partial_t^2 u_t - |\bar{\partial}_D(\partial_{\bar{t}} u)|_{\omega_{u_t}}^2) \frac{\omega_{u_t}^{n-1}}{(n-1)!} = \int_D |V_N|_{\omega_{u_t}}^2 \frac{\omega_{u_t}^{n-1}}{(n-1)!} (\geq 0)$$

where  $V_N$  denotes the component of  $V_t$  normal to  $D$  wrt  $\omega_t$  and where we have used the geodesic equation 4.6 in the last step. The proof is now concluded by invoking the decomposition formula for  $\mathcal{K}_\theta$  from the previous proposition.  $\square$

As shown by Bando-Mabuchi [3] any Kähler-Einstein metric minimizes  $\mathcal{K}_\theta$ . Here we note that the corresponding property holds in the (possibly singular) twisted setting for any *positive* current  $\theta$ :

**Proposition 4.3.** *Let  $\theta \geq 0$  be a positive current and  $u \in \mathcal{E}^1(X, \omega)$  a solution to equation 4.4, Then  $u$  minimizes the functional  $\mathcal{K}_\theta$  on  $\mathcal{E}(X, \omega)$ .*

*Proof.* By Theorem 3.4 it will be enough to prove that  $u$  minimizes the corresponding twisted Ding functional  $-\mathcal{G}_\theta$ . But  $-\mathcal{G}_\theta$  is convex along  $C^0$ -geodesics [13] and hence it is minimized on any critical point  $u$ .  $\square$

In the case when  $\theta \geq 0$  is smooth Stoppa [77] deduced the previous proposition from the geodesic convexity of  $\mathcal{K}_\theta$ , combined with the deep regularity theory for  $C^{1,1}$ -geodesics of Chen-Tian (in the more general setting of twisted constant scalar curvature metrics).

**4.2. Convergence of the Calabi flow (proof of theorem 1.6).** In this section we consider for simplicity the un-twisted case, i.e.  $\theta = 0$  (see Remark 4.5 below for the twisted case). First recall that the *Mabuchi-Semmes-Donaldson Riemannian metric*  $g$  on  $\mathcal{H}(X, \omega)$  is defined by first identifying the tangent space of  $\mathcal{H}(X, \omega) \subset \mathcal{C}^\infty(X)$  at the point  $u$  with  $\mathcal{C}^\infty(X)$  and then letting

$$g(v, v)|_u := \int_X v^2(\omega_u)^n / n!$$

We denote by  $d(\cdot, \cdot)$  the corresponding distance function on  $\mathcal{H}(X, \omega)$ . It follows directly from the variational definition of the Mabuchi's  $K$ -energy functional  $\mathcal{K}$  (see Proposition 4.1) that its gradient on  $(\mathcal{H}(X, \omega), g)$  is given by

$$\nabla \mathcal{K}|_u = -(R_{\omega_u} - R),$$

where  $R_{\omega_u}$  denotes the scalar curvature of the Kähler metric  $\omega_u$  and  $R$  its average, which is an invariant of the class  $[\omega]$ . The *Calabi functional* on  $\mathcal{H}(X, \omega)$  may be defined as the squared norm of  $\nabla \mathcal{K}$ , i.e.

$$Ca(u) := \int_X (R_{\omega_u} - R)^2 \omega_u^n / n!,$$

We let  $u_t$  evolve according to the Calabi flow on the level of Kähler potentials, i.e.

$$(4.7) \quad \frac{du_t}{dt} = (R_{\omega_{u_t}} - R)(= \nabla \mathcal{K}|_{u_t})$$

Before turning to the proof of Theorem 1.6 we recall the result of Tian [86] saying that if  $H^0(TX) = \{0\}$  then  $X$  admits a Kähler-Einstein metric iff the functional  $\mathcal{K}$  is proper (wrt energy); compare section 3.2.2. By Cor 1.5 and the uniqueness of the Kähler-Einstein metric under the assumptions above [3] it will be enough to prove that

$$(4.8) \quad \lim_{t \rightarrow \infty} \mathcal{K}(u_t) = \inf_{\mathcal{H}(X, \omega)} \mathcal{K} > -\infty$$

To this end first we first recall that following inequality of Chen [28]:

$$(4.9) \quad \mathcal{K}(u) - \mathcal{K}(v) \leq d(u, v)Ca(u)^{1/2}$$

Moreover, as shown by Calabi-Chen (see [21])  $d$  is decreasing under the Calabi flow and hence

$$(4.10) \quad d(u_t, v_t) \leq d(u_0, v_0)$$

for  $u_t$  and  $v_t$  evolving according to the Calabi flow 4.7. In particular, if we take  $v_0 := u_{KE}$  as a potential of a Kähler-Einstein metric  $\omega_{KE}$ , then  $v_t = v_0$  and hence

$$(4.11) \quad \mathcal{K}(u_t) - \mathcal{K}(u_{KE}) \leq d(u_0, u_{KE})Ca(u_t)^{1/2}$$

Next, observe that there is a sequence  $t_j$  such that

$$(4.12) \quad Ca(u_{t_j}) \rightarrow 0$$

as  $t_j \rightarrow \infty$ . Indeed, by the variational formula for  $\mathcal{K}$  we have

$$(4.13) \quad \frac{d\mathcal{K}(u_t)}{dt} = -Ca(u_t) \leq 0$$

Hence, if it would be the case that  $Ca(u_t) \geq \epsilon > 0$  as  $t \rightarrow \infty$  then this would force  $\mathcal{K}(u_t) \rightarrow -\infty$  as  $t \rightarrow \infty$  which contradicts the assumption that  $\mathcal{K}(u)$  be proper and in particular bounded from below. This proves the claim 4.12 and hence, by 4.11, we also get

$$(4.14) \quad \lim_{t_j \rightarrow \infty} \mathcal{K}(u_{t_j}) \leq \mathcal{K}(u_{KE}) = \inf_{\mathcal{H}(X, \omega)} \mathcal{K}$$

where the last property is a special case of Prop 4.3. Finally, by 4.13  $\mathcal{K}(u_t)$  is decreasing and hence the previous inequality implies the inequality 4.8, finishing the proof of the theorem.

*Remark 4.4.* The previous proof gave the weak convergence of  $\omega_{u_t}$ , which is equivalently to the  $L^1$ -convergence of the normalized potentials  $u_t - \sup u_t$ . But in fact the  $L^1$ -convergence holds for  $u_t$  (i.e. without normalising). Indeed, by the monotonicity and properness of  $\mathcal{K}$  we have that  $J_\omega(u_t) \leq C$ . Since,  $d\mathcal{E}_\omega(u_t)/dt = 0$  this means that  $\int u_t \omega^n \leq C'$ . But it follows from standard compactness arguments (for example used in [10]) that  $\{J_\omega \geq C\} \cap \{\int(\cdot)\omega^n \leq C'\}$  is relatively compact in  $PSH(X, \omega)$  and hence so is the set  $\{u_t\}$ , showing that there is no need to normalise  $u_t$ .

One final remark about the twisted case:

*Remark 4.5.* The previous proof admits a straight-forward generalization to the setting of twisted Kähler-Einstein metrics when  $\theta \geq 0$ , where  $R_\omega$  is replaced by the trace of the twisted Ricci curvature. Indeed, if  $\theta \geq 0$  the twisted functional  $\mathcal{K}_\theta$  is still geodesically convex (see Prop 4.2) which at least formally implies 4.9 and 4.10. Hence the Hessian of  $\mathcal{K}_\theta$  (defined wrt the metric  $g$  above) is a semi-positive Hermitian operator which implies that the corresponding flow decreases the length of any initial curve and is hence distance decreasing (compare the proofs in [21] and [27]). The estimate 4.11 is more involved as it requires a notion of *weak*  $C^{1,1}$ -geodesics, but the proof is a simple modification of the argument in [28].

**4.3. Alpha invariants and Nadel sheaves (proof of Corollary 1.7).** The first point of the corollary is a direct consequence of Theorem 1.3 applied to the twisted Kähler-Einstein setting. Next, we show how the proof can be refined so as to give a proof of the second point in the corollary. After scaling we may assume that  $\gamma = 1$ . Let  $(u_j)$  be an asymptotic minimizer of  $\mathcal{K}_\theta$  such that  $u_j \rightarrow u_\infty$  in  $L^1$  (by weak compactness such an  $u_\infty$  always exists). If the second alternative in the statement of Cor 1.7 does not hold then there is  $t > \frac{n}{n+1}$  such that  $\int e^{-tu_\infty} dV < \infty$ . But then it follows from the semi-continuity result of Demailly-Kollar [35] (see Thm 6.1 in the appendix) that  $\int e^{-tu_j} dV \leq C < \infty$  after perhaps replacing  $t$  with any strictly smaller number. In the notation of the proof given in section 3.4.1 this means that  $\mathcal{L}_t^-(u_j) > -C'$  and hence repeating that proof word for word shows that

$$(4.15) \quad \mathcal{K}_\theta(u_j) \geq J(u_j)/C - C$$

for some constant  $C$ . Finally Theorem 3.8 shows that  $u$  is a minimizer for  $\mathcal{K}_\theta$  and satisfies the twisted Kähler-Einstein equation.

*Remark 4.6.* The second point in Cor 1.7 generalizes Nadel's original result [68]; letting  $T$  be the sup over all positive  $t$  such that the equations 1.6 have a solution, Nadel shows (see also the simplifications in [35]) that either  $T \geq 1$  and the potential  $u_t$  of  $\omega_t$  converges to a Kähler-Einstein metric or there is sequence  $t_j \rightarrow T$  such  $u_{t_j} \rightarrow u_T$  for  $u_T (= u_\infty)$  as in the second point of Cor 1.7. To see that this is a special case of Cor 1.7 we argue as above; if the second alternative does not hold then one checks that  $u_t$  is an asymptotic minimizing sequence for  $\mathcal{K}_{\theta_T}$  where  $\theta_t := (1-t)\omega$  (see below) and hence we may apply the second point in Cor 1.5 (with  $t = \gamma \leq 1$  and  $\theta = \theta_T$ ) to deduce that  $u_{t_j} \rightarrow u_T$ , where  $\omega_{u_T}$  solves the twisted Kähler-Einstein equation for  $\theta = \theta_T$ . But then it follows from the definition of  $T$  that  $T \geq 1$  and hence  $\omega_{u_1}$  is a Kähler-Einstein metric proving Nadel's result. Finally, the asymptotic minimizing property above is shown as follows: as is well-known  $\mathcal{K}_0(u_t)$  is decreasing in  $t$  and hence  $J(u_t) \leq C$  (by 4.15). But since  $\theta_t \geq 0$   $u_t$  is the absolute minimizer of  $\mathcal{K}_{\theta_t}$  (see the end of Remark 4.5) one deduces (also using  $J(u_t) \leq C$ ) the desired asymptotic minimizing property (by the same argument used in Step 2 in the proof of Cor 5.2)

## 5. LOG FANO MANIFOLDS AND DONALDSON'S EQUATION

It will be convenient to use the formalism of metrics on line bundles, which we briefly recall (for further details see the book [37]). A cohomology class  $T$  in  $H^{1,1}(X, \mathbb{R})$  is *integral*, i.e.  $T \in H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Z})$  precisely when there is a holomorphic line bundle  $L \rightarrow X$  such that  $T = c_1(L)$ , the first Chern class of  $L$ . We will use the weight notation for metrics on line bundle: a metric on  $L$  may be represented by  $e^{-\phi_0}$  (the point-wise norm of the trivializing section) where the local function  $\phi_0$  is called the *weight* of the metric. The normalized curvature form of the metric is then the globally well-defined form, locally expressed as  $dd^c \phi \in c_1(L)$ . Conversely, any form  $\omega$  in the class  $c_1(L)$  may be written as  $\omega = dd^c \phi$  for a weight on  $L$  which is uniquely determined mod  $\mathbb{R}$ . Hence, upon fixing a reference weight  $\phi_0$  on  $L$  and letting  $\omega_0 := dd^c \phi_0$  one gets a bijection

$$(5.1) \quad PSH(X, L) \longleftrightarrow PSH(X, \omega_0), \phi \mapsto u = \phi - \phi_0 \quad (\implies \omega_u = dd^c \phi)$$

between the space of all positively curved singular weights  $\phi$  on  $L$  and  $PSH(X, \omega_0)$ .

We now specialize the setting of twisted Kähler-Einstein to the case when the twisting class  $[\theta]$  is integral and write  $\theta = dd^c \phi_0$  for a weight  $\phi_0$  on a line bundle

$L \rightarrow X$ . Then the twisted Kähler-Einstein equation is equivalent to the following equation for the weight  $\psi$  of a metric on the  $\mathbb{R}$ -line bundle  $\frac{1}{\beta}(K_X + L)$  :

$$(5.2) \quad (dd^c \psi)^n / n! = e^{\beta\psi - \phi_0} idz_1 \wedge d\bar{z}_1 \wedge \dots idz_n \wedge d\bar{z}_n$$

In this section we will be mainly interested in the most challenging case when  $\beta < 0$  (i.e. the twisted Ricci curvature is positive) and consider the singular case when the twisting form  $\theta$  is a linear combination of the integration currents along codimension one analytic subvarieties in  $X$ , i.e.

$$\theta := \sum c_i \delta_{D_i},$$

where  $D_i$  is an irreducible subvariety in  $X$ . In other words,  $D_i$  is an irreducible effective divisor and we write

$$\Delta := \sum c_i D_i,$$

for the corresponding  $\mathbb{R}$ -divisor on  $X$  (abusing notation slightly we will also denote its support by  $\Delta$ ). Equivalently, writing  $D_i := \{s_i = 0\}$  for some holomorphic section  $s_i$  of a line bundle  $L_{D_i} \rightarrow X$  this means that

$$(5.3) \quad \phi_0 := \psi_\Delta := \sum_i c_i \log |s_i|^2$$

defines a singular weight on the corresponding  $\mathbb{R}$ -line bundle  $L_\Delta$ . We will assume that the  $D_i$ 's are distinct and smooth with simple normal crossings (i.e. there are local coordinates where  $D_i = \{z_{m(i)} = 0\}$ ) and  $c_i < 1$  for all  $i$  to make sure that the rhs in equation 5.2 is locally integrable. In the language of the minimal model program in algebraic geometry this means that the *log pair*  $(X, \Delta)$  is *sub klt* (sub Kawamata Log Terminal) and *klt* if moreover  $c_i \geq 0$ . The measure  $\mu_0$  in formula 4.5 is now well-defined and may be written as

$$(5.4) \quad \mu_0 = \mu_\Delta := \prod_i |s_i|^{-2c_i} e^{-\phi_\Delta} dV$$

for some volume form  $dV$  on  $X$  and smooth metric  $e^{-\phi_\Delta}$  on  $L_\Delta$

Before continuing we briefly point out that in the case when  $c_i$  is positive and  $c_i = 1 - 1/m_i$ , where the  $m_i$ 's are integers such that  $m_i$  and  $m_j$  are coprime if  $D_i \cap D_j \neq \emptyset$  the pair  $(X, \Delta)$  determines an *orbifold* with codimension one stabilizers  $\mathbb{Z}/m_i\mathbb{Z}$  along  $D_i$  [18, 50, 80]. Moreover, the measure  $\mu_0$  is then a volume form in the orbifold sense (i.e. it lifts to a smooth volume form in the local coverings defined by the orbifold).

By Prop 3.1 any finite energy solution  $\psi$  of equation 5.2 is locally bounded. Its curvature current  $\omega (= dd^c \psi) \in \beta c_1(K_X + L_\Delta)$  satisfies the following singular Kähler-Einstein equation (to simplify the notation we set  $\beta = -1$ ) :

$$(5.5) \quad \text{Ric } \omega = \omega + \delta_\Delta$$

in the sense of currents (where  $\text{Ric } \omega$  now denotes the curvature current of the induced singular metric on  $-K_X$ ). Conversely, we have the following simple

**Proposition 5.1.** *Let  $\omega$  be a smooth Kähler-Einstein metric (with  $\text{Ric } \omega = -\beta\omega$ ) on  $X - \Delta$  such that*

$$\frac{1}{C} \prod_i |s_i|^{-2c_i} e^{-\phi_\Delta} \omega_0^n \leq \omega^n \leq C \prod_i |s_i|^{-2c_i} e^{-\phi_\Delta} \omega_0^n$$

*in terms of a smooth metric  $e^{-\phi_\Delta}$  on  $L_\Delta$  and Kähler form  $\omega_0$  on  $X$ . Then  $1_{X-\Delta}\omega$  defines a closed positive current in  $\beta c_1(K_X + L_\Delta)$  which is the curvature current*

of a locally bounded metric on  $\beta(K_X + L_\Delta)$  whose weight  $\psi$  satisfies the global Monge-Ampère equation 5.2 on  $X$ , weakly.

*Proof.* The local representations  $-\log(\omega^n)$  on  $X - \Delta$  define a singular weight (i.e. in  $L^1_{loc}$ ) on  $-K_X$  such that

$$-\log(\omega^n) =: \psi + \psi_\Delta,$$

where, by the growth assumptions on  $\omega^n$   $\psi$  defines a singular weight on  $-(K_X + L_\Delta)$  which is smooth on  $X - \Delta$  and locally bounded as a global weight, i.e. if we introduce the  $\omega_0$ -psh function  $u := \psi - \psi_0$  for a fixed smooth positively curved weight  $\psi_0$  on  $-(K_X + L_\Delta)$  (as in the isomorphism 5.1) then  $|u| \leq C$  on  $X - \Delta$  and hence  $u$  extends to a unique element in  $L^\infty(X) \cap PSH(X, \omega_0)$ . Moreover, since by assumption  $\omega$  is Kähler-Einstein on  $X - \Delta$  applying  $dd^c$  to the previous equation (i.e. computing the corresponding curvature currents) gives

$$dd^c \psi = \omega \quad (\text{i.e. } \omega_u = \omega)$$

on  $X - \Delta$ . Combining the last two equations shows that the Monge-Ampère equation 5.2 is satisfied on the complement of  $\Delta$  which, expressed in terms of  $u$ , means that

$$\omega_u^n = e^{-u} e^{-h_{\omega_0, \delta_\Delta}} \omega_0^n \quad \text{on } X - \Delta$$

where  $h_{\omega_0, \delta_\Delta}$  is a (singular) twisted Ricci potential on  $X$ . Finally, since  $u \in L^\infty(X) \cap PSH(X, \omega_0)$  and hence its *global* Monge-Ampère measure  $\omega_u^n$  does not charge the pluripolar set  $\Delta$  the previous equation in fact holds on all of  $X$  and for the same reason  $\omega_u = 1_{X-\Delta} \omega_u$  in the sense of currents.  $\square$

The main geometric situation where the volume form asymptotics in the previous propositions hold is when  $\Delta = (1 - t)D$ , where  $t > 0$ ,  $D$  is a smooth divisor and  $\omega$  is a Kähler metric with cone angles  $2\pi t$  in the directions transverse to  $D$ , where

$$(5.6) \quad \frac{1}{C} \omega_{cone} \leq \omega \leq C \omega_{cone}, \quad \omega_{cone} := (|z_1|^{-2(1-t)} dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + \dots + dz_n \wedge d\bar{z}_n)$$

in local coordinates close to  $D = \{z_1 = 0\}$ . In particular, this is the case when  $t = 1/m$  and  $\omega$  is an orbifold Kähler metric for  $(X, \Delta)$ , just using that  $|z_1|^{-2(1-1/m)} dz_1 \wedge d\bar{z}_1 = dw \wedge d\bar{w}$ , if  $z = w^m$  (compare the discussion in [80]).

We next define the  $\alpha$ -invariant of the pair  $(X, \Delta)$  by

$$\alpha(X, \Delta) := \alpha(-c_1(K_X + \Delta), \mu_\Delta).$$

In the orbifold case  $\alpha(X, \Delta)$  coincides with the  $\alpha$ -invariant (i.e. the log canonical threshold) of an orbifold associated to  $(X, \Delta)$  and was studied by Demailly-Kollar [35].

Applying Theorem 1.3 combined with Kolodziej's regularity theorem (just like in the proof of Theorem 3.2) now gives the first statement in the following corollary concerning continuous solutions to equation 5.2. To obtain smoothness on  $X - \Delta$  we will show that the solution is the limit of smooth solutions to the twisted Kähler-Einstein equations obtained by replacing the current  $\Delta$  with a sequence of regularizations.

**Corollary 5.2.** *Let  $(X, \Delta)$  be a pair as above and assume that*

$$(5.7) \quad \alpha(X, \Delta) > \frac{n}{n+1}$$

*Then there is a Hölder continuous solution to equation 5.2 and its curvature current  $\omega$  satisfies equation 5.5. Moreover, if  $(X, \Delta)$  is a klt pair, i.e. the current  $\delta_\Delta$  is*

positive, then  $\omega$  may be taken to be a smooth Kähler-Einstein metric on  $X - \Delta$  and globally on  $X$  it is a Kähler current, i.e. there is a Kähler form  $\omega_0$  on  $X$  such that  $\omega \geq \omega_0$  on  $X$ . When  $(X, \Delta)$  defines an orbifold there is a solution  $\omega$  which is smooth in the orbifold sense.

*Proof.* The existence of a Hölder continuous solution of 5.2 is a special case of Theorem 1.3 combined with Kolodziej's result (just as in the proof of Prop 3.1).

*Higher order regularity when  $\theta := \delta_\Delta \geq 0$ :*

We start by fixing a smooth weight  $\psi_0$  on  $-(K_X + L_\Delta)$  with curvature form  $\omega_0 > 0$ . Let  $\psi_\Delta^{(j)}$  be a sequence of strictly positively curved smooth weights on  $-(K_X + L_\Delta)$  decreasing to  $\psi_\Delta$  (for example we may take  $\psi_\Delta^{(j)} := \log(\sum c_i |s_i|^2 + e^{\psi_0}/j)$ ). Hence,  $\theta_j := dd^c \psi_\Delta^{(j)}$  are Kähler forms in  $-c_1(K_X + L_\Delta)$  and we take  $u_j =: \psi_j - \psi_0 \in PSH(X, \omega_0)$  to be a sequence of minimizers, normalized so that  $\sup_X u_j = 0$ , of the corresponding twisted Mabuchi functionals  $\mathcal{K}_{\theta_j}$ . Since  $\alpha(-c_1(K_X + L_\Delta)) \geq \alpha((-c_1(K_X + L_\Delta), \mu_\Delta)) (= \alpha(X, \Delta)) > n/(n+1)$  such minimizers exist and are smooth according to Thm 1.7 and satisfy

$$(5.8) \quad \frac{\omega_{u_j}^n}{n!V} = \frac{e^{-u_j} \mu_\Delta^{(j)}}{\int_X e^{-u_j} \mu_\Delta^{(j)}}; \quad \text{Ric } \omega_{u_j} = \omega_{u_j} + \theta_j$$

where  $\mu_\Delta^{(j)}$  are volume forms on  $X$  increasing to the measure  $\mu_\Delta$ . We may (after perhaps passing to a subsequence) assume that  $u_j \rightarrow u_\infty$  in  $L^1(X)$ .

*Step 1:*  $J_\omega(u_j) \leq C$ ,  $\int e^{-(1+\epsilon)u_j} \mu_\Delta \leq C$

It follows directly from the assumption 5.7 that there is a  $t > n/(n+1)$  such that

$$\int e^{-tu_j} \mu_\Delta^{(j)} \leq \int e^{-tu_j} \mu_\Delta \leq C_t$$

Repeating the argument in the proof given in section 3.4.1 word for word hence gives the coercivity estimate

$$\mathcal{K}_{\theta_j}(u_j) \geq J_{\omega_0}(u_j)/C'_t - C'_t$$

Moreover, the minimization property of  $u_j$  implies that  $\mathcal{K}_{\theta_j}(u_j)$  is uniformly bounded from above (indeed,  $\mathcal{K}_{\theta_j}(u_j) \leq \mathcal{K}_{\theta_j}(0) := -F_{\mu_\Delta^{(j)}}(\frac{\omega_0^n}{n!V}) \rightarrow -F_{\mu_\Delta}(\frac{\omega_0^n}{n!V})$  by the monotone convergence theorem). But then the previous inequality forces

$$J_{\omega_0}(u_j) \leq C$$

Next, the assumption 5.7 also shows that  $\mathcal{K}_\theta$ , where  $\theta = \delta_\Delta$ , is coercive and hence  $F_{\beta, \mu_\Delta}$  is bounded from above for some  $\beta = -(1+\epsilon)$  where  $\epsilon > 0$  and thus by Theorem 3.4 so is  $G_{\beta, \mu_\Delta}$ , i.e.

$$(5.9) \quad \frac{1}{1+\epsilon} \log \int e^{-(1+\epsilon)u_j} \mu_\Delta \leq J(u_j) + C' \leq C + C'$$

*Step 2:* The sequence  $u_j$  is an asymptotic minimizer of  $\mathcal{K}_\theta$  (and hence  $\omega_{u_j} \rightarrow \omega_{u_\infty}$  solving equation 5.5)

We fix an arbitrary Kähler potential  $u$  in  $\mathcal{H}(X, \omega_0)$ . Since by assumption  $u_j$  is a minimizer of  $\mathcal{K}_{\theta_j}$ , and hence by Theorem 3.4 also a maximizer of  $\mathcal{G}_{\mu_\Delta^{(j)}}$  we have

$$\mathcal{G}_{\mu_\Delta^{(j)}}(u_j) \geq \mathcal{G}_{\mu_\Delta^{(j)}}(u) \rightarrow \mathcal{G}_{\mu_\Delta}(u)$$



using the monotone convergence theorem. By the upper bound 5.9 we have, according to Lemma 3.6, that

$$\log \int e^{-u_j} \mu_{\Delta}^{(j)} \leq \log \int e^{-u_j} \mu_{\Delta} \rightarrow \log \int e^{-u_{\infty}} \mu_{\Delta}$$

and hence it follows from the upper semi-continuity of  $\mathcal{E}_{\omega_0}$  that

$$\mathcal{G}_{\mu_{\Delta}}(u_{\infty}) \geq \limsup_{j \rightarrow \infty} \mathcal{G}_{\mu_{\Delta}^{(j)}}(u_j) \geq \mathcal{G}_{\mu_{\Delta}^{(j)}}(u) \rightarrow \mathcal{G}_{\mu_{\Delta}}(u)$$

By Theorems 1.2 and 3.4 this finishes the proof.

*Step 3:*  $\sup_X |u_j| \leq C$

By the first equation in 5.8 and step 1 above we have that  $\omega_{u_j}^n / \omega_0^n$  is uniformly bounded in  $L^{(1+\epsilon)}(X, \omega_0^n)$  and hence Kolodziej's theorem [58] gives the desired  $C^0$ -bound

*Step 4:* (a)  $\omega_{u_j} \geq \frac{1}{C} \omega_0$  on  $X$ , (b)  $\sup_K |\omega_{u_j}|_{\omega_0} \leq C_K$  on  $K \subset\subset X - \Delta$

First observe that since  $\theta_j \geq 0$  equation 5.8 shows that the Ricci curvature of  $\omega_{u_j}$  is uniformly bounded from below on  $X$  (by a positive constant, but a negative constant would also be fine for the following argument). Combined with the uniform bound on  $u_j$  in the previous step it follows from an argument in [4] which is a variant of the usual Aubin-Yau Laplacian estimate [2, 93] that (a) holds (the author learned the argument from [73] where it used to handle another situation where  $\text{Ric } \omega_{u_j}$  is uniformly bounded from below). We next recall the argument: it follows directly from the Chern-Lu (in)equality that

$$\Delta_{\omega_{u_j}}(\log(\text{Tr}_{\omega_{u_j}} \omega_0)) \geq -C(\text{Tr}_{\omega_{u_j}} \omega_0)$$

using that there is a positive lower bound of the Ricci curvature of  $\omega_{u_j}$  and where  $C$  is the upper bound of the bisectional curvature of  $\omega_0$ . Since,  $\text{Ric } \omega_{u_j} \geq \omega_{u_j}$  it follows that there is a constant  $C$  independent of  $u_j$  such that, setting  $v_j := \text{Tr}_{\omega_{u_j}} \omega_0$ , we have

$$(5.10) \quad \Delta_{\omega_{u_j}}(\log v_j - (C+1)u_j) \geq -(C+1)n + v_j$$

Evaluating the inequality above at a point where  $\log v_j - (C+1)u_j$  attains its maximum (so that the lhs above is non-positive) and using that  $u_j$  is, by Step 3 above, uniformly bounded gives an upper bound on  $\sup_X v_j$  which implies the desired lower bound on  $\omega_{u_j}$ . Next, by equation 5.8 and Step 3 above we have that  $\omega_{u_j}^n / \omega_0^n$  is uniformly bounded from above on any fixed compact set  $K$  in  $X - \text{supp } \Delta$  which finishes the proof of Step 4.

*Step 5:*  $\exists \alpha > 0 : \|u_j\|_{C^{2,\alpha}(K)} \leq C$  on  $K \subset\subset X - \Delta$

Given the previous estimates which, in particular, show that  $\|u_j\|_{L^{\infty}(K)} \leq C$ ,  $\|\Delta_{\omega_0} u_j\|_{L^{\infty}(K)} \leq C$  and  $MA(u_j) \geq 1/C$ , step 5 follows from a complex version of the Evans-Krylov-Trudinger theory for local non-linear elliptic equations (see Thm 5.1 in [16]).

Finally, using the standard linear elliptic local (Schauder) estimates and bootstrapping shows that  $\|u_j\|_{C^{p,\alpha}(K)} \leq C_p$  for any  $p > 0$  and hence (after perhaps passing to a subsequence) it follows that  $u_j \rightarrow u_{\infty}$  in the  $C^{\infty}$ -topology on compacts on  $X - \Delta$ . In particular, this shows that  $u_{\infty}$  is smooth on  $X - \Delta$ .  $\square$

It may be worth pointing out that the variational part of the proof above (i.e. Step 2) is not really needed as the rest of the argument anyway produces a bounded function  $u_{\infty}$  on  $X$  satisfying the limiting version of the Monge-Ampère equation 5.8 on  $X - \Delta$  and hence everywhere since the support of  $\Delta$  is a pluripolar set

(see Prop 5.1). But one of the main virtues of the variational approach is that it gives the convergence of *any* sequence  $u_j$  which is an asymptotic maximizer of the corresponding twisted Mabuchi functional (under the usual properness assumption). In particular, the previous corollary can be made more precise giving a singular variant (i.e applied to  $\theta = \delta_\Delta$ ) of the second point of Cor 1.7 obtained by replacing the volume form  $dV$  used in the exponential integral of  $u_\infty$  with the measure  $\mu_\Delta$ .

In the orbifold case Cor 5.2 is essentially due to Demailly-Kollar who obtained a solution  $\omega$  which is a Kähler metric in the orbifold sense [35]. Strictly speaking the results in [35] were formulated in the classical orbifold setting of stabilizers of codimension  $> 1$  (then  $X$  has quotient singularities), but the same arguments are valid in the codimension one case (as pointed out in [18]).

**5.1. Donaldson's equation and the proof of Theorem 1.8.** The existence of solutions to Donaldson's equation 1.10 will be deduced from the criterion in Cor 5.2 concerning the alpha invariant of a pair  $(X, \Delta)$  and the following lower bound on such invariants in the particular setting of Donaldson's equation. One of the ingredients in the proof is a simple extension to pairs of the well-known identification between alpha-invariants and log canonical thresholds (see the appendix).

**Proposition 5.3.** *Let  $L$  be an ample line bundle over  $X$  and  $s$  a holomorphic section of  $L$  such that  $D := \{s = 0\}$  is an irreducible smooth divisor. Then*

$$(5.11) \quad \alpha(L, \mu_{(1-\gamma)D}) \geq \min\{\gamma, \alpha(L), \alpha((L|_D))\}$$

*Proof.* By Proposition 6.3 in the appendix it will be enough to prove that if  $s_m \in H^0(mL)$  then  $-t(\frac{1}{m} \log |s_m|^2)$  is locally integrable wrt  $\frac{1}{|s|^{2(1-\gamma)}} dV$  for any fixed  $t$  strictly smaller than the rhs in 5.11. To this end we first recall that following inequality, which is an immediate consequence of the Ohsawa-Takegoshi extension theorem (see Thm 2.1 in [35] and references therein): If  $u \in PSH(\Omega)$  such that  $u$  is not identically  $-\infty$  on the smooth connected complex submanifold  $\{s = 0\} \subset \Omega \subset \mathbb{C}^n$  then, for  $\delta > 0$ ,

$$(5.12) \quad \int_U e^{-u} \frac{1}{|s|^{2(1-\delta)}} dV_n \leq C_\delta \int_{\{s=0\}} e^{-u} dV_{n-1}$$

on some neighborhood  $U \subset \Omega$  containing  $\{s = 0\}$  (depending on  $u$ ). Now take  $s_m \in H^0(X, mL)$  and decomposes  $s_m = s^{\otimes l} \otimes s'$  where  $l \leq m$  and  $s' \in H^0((m-l)L)$  does not vanish identically on  $D := \{s = 0\}$  unless  $l = m$ . In the case when  $m = l$  the integral  $I_t$  is clearly finite as long as  $t < \gamma$ . Otherwise the bound  $l/m < 1$  translates to

$$\begin{aligned} e^{-t \frac{1}{m} \log |s_m|^2} \frac{1}{|s|^{2(1-\gamma)}} &= e^{-t(\frac{1}{m} \log |s|^2)} e^{-t(\frac{m-l}{m}) \frac{1}{m-l} \log |s'|^2} \frac{1}{|s|^{2(1-\gamma)}} \leq \\ &\leq e^{-t \frac{1}{m-l} \log |s'|^2} \frac{1}{|s|^{2(1-\delta)}} \end{aligned}$$

for any fixed  $t \leq \gamma - \delta$ . Since,  $\frac{1}{m-l} \log |s'|^2$  is a psh weight on  $L$  the inequality 5.12 gives that the function  $e^{-t \frac{1}{m} \log |s_m|^2} \frac{1}{|s|^{2(1-\delta)}}$  is locally integrable in a neighborhood of  $\{s = 0\}$  as long as  $t \leq \inf\{\gamma, \alpha(L|_D)\} - \delta$ . Moreover, on the complement of a neighborhood of  $\{s = 0\} \subset X$  the factor  $\frac{1}{|s|^{2(1-\gamma)}}$  is bounded and hence  $e^{-t \frac{1}{m} \log |s_m|^2} \frac{1}{|s|^{2(1-\delta)}}$  is locally integrable there as long as  $t < \alpha(L)$ . All in all, this means that the integral  $I_t(\frac{1}{m} \log |s_m|^2)$  is finite if  $t \leq \min\{\gamma, \alpha(L), \alpha((L|_D))\} - \delta$ .  
5.11. □

Before continuing with the proof of Theorem 1.8 we make two remarks. First we note that it follows immediately from Hölder's inequality that

$$\alpha(L, \mu_{(1-\gamma)D}) \geq \gamma \alpha(L)$$

But the point with the previous proposition is that it will allow us to deduce the existence of a solution to Donaldson's equation for  $\gamma$  sufficiently small without assuming that the classical  $\alpha$ -invariant is sufficiently big, i.e. without assuming that  $\alpha(L) > n/(n+1)$ . Secondly, the lower bound in the previous proposition should be compared with the trivial upper bound  $\alpha(L, \mu_{(1-\gamma)D}) \leq \min\{\gamma, \alpha(L)\}$  (just take  $\psi := \log |s|^2$ ). In the one dimensional case when  $\psi_\Delta$  is the singular weight defined by a sub klt divisor (formula 5.3) and  $V := \deg L = 1$  a slight modification of the proof above gives

$$(5.13) \quad \alpha(L, \mu_\Delta) = \min_i \{\alpha(L, 1 - c_i)\} = \min_i \{1, 1 - c_i\}$$

(this also follows from the argument in example 3.10 since  $\exp(-t(g_{x_0}))$  is integrable wrt  $\mu_\Delta$  iff  $t < \min_i \{1, 1 - c_i\}$ ).

*Existence:*

By a simple rescaled version of Corollary 5.2 there is a solution if

$$\alpha(-(K_X), \mu_{(1-\gamma)D}) > \gamma \frac{n}{n+1}$$

and by the previous Proposition 5.3 this inequality is clearly satisfied if  $\gamma < \Gamma := \frac{n+1}{n} \min \{ \alpha(-K_X), \alpha((-K_X)|_D) \}$ .

*Uniqueness:*

According to Berndtsson's very recent generalized Bando-Mabuchi uniqueness theorem [13] there is a unique solution of Donaldson's equation 1.10 unless there is a non-trivial holomorphic vector field  $V$  on  $X$  which is tangent to  $D$  (formally this is a consequence of the strict convexity in Prop 4.2, but the problem is the non-existence of bona fide geodesics connecting two critical points). Next, we give a direct argument (which does not rely on the previous existence result) showing that such a  $V$  does not exist. Assume to get a contradiction that  $V$  as above does exist and take  $\gamma$  sufficiently small (so that  $0 < \gamma < \Gamma$ ). As shown above  $\mathcal{K}_{(1-\gamma)D}$  is proper wrt energy (since the condition on the alpha-invariant of  $(X, (1-\gamma)D)$  is satisfied). Hence it will, to reach a contradiction, be enough to find a curve  $u_t$  such that  $J_\omega(u_t)$  tends to infinity, but  $\mathcal{K}_{(1-\gamma)D}(u_t)$  does not. To this end we let  $u_t$  be defined by  $u_t := -\log(h_t/h)$  where  $h_0$  is a fixed metric on  $-K_X$  with curvature form equal to the Kähler metric  $\omega$  and  $h_t := F_t^* h_0$  where  $F_t$  denotes the lift to  $-K_X$  of the flow defined by  $V$ . Then  $u_t$  satisfies the geodesic equation 4.6, where  $V_t$  coincides with  $V$ , the given holomorphic vector field (compare [41]). Setting  $J(t) := J_{\omega_0}(u_t)$  a direct calculation gives

$$\frac{d^2 J(t)}{dt^2} = \int_X \partial_t^2 u_t \frac{\omega^n}{n!} = \int_{X-D} |V_t|_{\omega_t}^2 \frac{\omega^n}{n!} > 0$$

if  $V$  is non-trivial and hence  $J(t) \rightarrow \infty$  as  $|t| \rightarrow \infty$ . Finally, Prop 4.2 implies that  $\mathcal{K}_{(1-\gamma)D}(u_t)$  is affine wrt  $t$ . Hence, the limit of  $\mathcal{K}_{(1-\gamma)D}(u_t)$  is bounded from above when either  $t \rightarrow \infty$  or  $t \rightarrow -\infty$  giving the desired contradiction.

*Regularity of the curve  $\gamma \mapsto \omega_\gamma$*

Fix  $\gamma = \gamma_0 \in ]0, \Gamma]$ . Since the (normalized) potential  $u_\gamma$  of the Kähler-Einstein current  $\omega_\gamma$  maximizes the functional  $\mathcal{G}_\gamma := \mathcal{G}_{-\gamma, (1-\gamma)D}$  it is not hard to check that  $\mathcal{G}_{\gamma_0}(u_t)$  converges, when  $\gamma \rightarrow \gamma_0$ , to the supremum of  $\mathcal{G}_{\gamma_0}$  (this is similar to the proof of step 2 in the proof of Theorem 1.8) and hence it follows, just like in the

Step 2 in the proof of Cor 5.2, that any limit point in the  $L^1$ -closure of  $\{u_\gamma\}$  is a maximizer of  $\mathcal{G}_{t_0}$ . By the uniqueness in the previous point this means that  $\omega_\gamma \rightarrow \omega_{\gamma_0}$  in the sense of currents. Finally, to prove the stronger continuity it is enough to show that, for any positive integer  $m$ , the partial derivatives of  $u_\gamma$  total order  $m$  are uniformly bounded on a given compact subset  $K$  in  $X - D$  with a constant which is independent of  $\gamma$ . But this follows from writing  $u_\gamma$  as the limit of  $u_\gamma^{(j)}$  ( $=: u_j$ ) where  $u_j$  was defined in the proof of the previous corollary (where higher order estimates were obtained with constants which are clearly independent of  $\gamma$ ).

*Remark 5.4.* In the case of a Riemann surface Cor 5.2 combined with the simple identity 5.13 gives a new proof of Troyanov's existence result for metrics with constant positive curvature and conical singularities ([91], Thm C). Note that the proof in [91] was also variational, but our derivation of the corresponding Moser-Trudinger inequality is new (the proof in [91] uses weighted Sobolev inequalities).

## 6. APPENDIX: $\alpha$ -INVARIANTS (LOG CANONICAL THRESHOLDS) FOR PAIRS

In this appendix we will extend the results of Demailly in [36] concerning  $\alpha$ -invariants of Kähler classes to a more singular setting and in particular to the setting of klt pairs considered in section 5. The proofs are almost the same as in the usual setting, but have been included as a courtesy to the reader. We will denote by  $PSH(X, L)$  the space of all (possible singular) weights on  $L$  whose curvature is positive in the sense of currents, equipped with the  $L^1$ -topology.

Let  $L$  and  $\Delta$  be holomorphic line bundles over a compact Kähler manifold  $X$  and fix smooth reference weights  $\phi_L$  and  $\phi_\Delta$  on  $L$  and  $\Delta$ , respectively. Let  $\psi_\Delta$  be a singular psh weight on  $\Delta$  such that the following measure is finite:

$$\mu_\Delta = e^{-(\psi_\Delta - \phi_\Delta)} dV$$

which we may assume is a probability measure for a suitable choice of volume form  $dV$  on  $X$  (the main case will be when the curvature current of  $\psi_\Delta$  is a klt divisor, as in section 5 and the same arguments apply to the sub klt case since the contribution from the negative coefficients  $c_i$  is bounded by a constant). For a fixed positive number  $t$  we consider the functional

$$I_t(\psi) := \int e^{-t(\psi - \phi_L)} \mu_\Delta$$

on the space  $PSH(X, L)$ . By definition  $\alpha(c_1 L, \mu_\Delta) :=$

$$:= \alpha(L, \mu_\Delta) := \sup \{t : I_t \text{ is bounded from above on } PSH(X, L) \cap \{\sup_X(\cdot - \phi_L) = 0\}\}$$

(which is in this case also can be expressed as a *relative critical exponent* in the terminology of [69]). Let us start by recalling the following fundamental local result from [35]

**Theorem 6.1.** (*Demailly-Kollar*). *Let  $K$  be a compact subset in a domain  $\Omega \subset \mathbb{C}^n$  and  $u \in PSH(\Omega)$ . Define  $c_K(u)$  as the sup over all  $c \geq 0$  such that  $e^{-cu}$  is integrable on some neighborhood of  $K$ . If  $u_j \rightarrow u$  in  $L^1(\Omega)$  where  $u_j \in PSH(\Omega)$ , then  $e^{-c u_j} \rightarrow e^{-c u}$  in  $L^1$  on some neighborhood of  $K$  for any  $c$  such that  $c < c_K(u)$ .*

Applying this theorem to the present global setting gives the following

**Corollary 6.2.** *If the functional  $I_{(t+\epsilon)}$  is finite on  $PSH(X, L)$  for some  $\epsilon > 0$  then  $I_t$  is continuous. As a consequence,*

- *For any fixed  $t < \alpha(L, \mu_\Delta)$  the functional  $I_t$  is continuous on  $PSH(X, L)$  (wrt the  $L^1$ -topology).*

- $\alpha(L, \mu_\Delta) := \sup \{t : I_t < \infty \text{ on } PSH(X, L)\}$

*Proof.* Take  $t$  and  $\epsilon > 0$  such that  $I_{t+\epsilon}$  is finite on  $PSH(X, L)$ . Assume that  $\psi_j \rightarrow \psi$  in  $PSH(X, L)$  and normalize so that  $\sup_X(\psi - \phi_L) = 0$ . Fix a point  $x$  and a neighborhood  $U$  of  $x$  where both  $L$  and  $L_\Delta$  are trivialized so that  $u_j := \psi + \psi_\Delta/t$  is represented by a local psh function. By assumption and the previous theorem we have, with  $f$  denoting the local smooth function  $f := t\phi_L + \phi_\Delta$ , that

$$\int_U e^{-t(\psi_j - \phi_L)} \mu_\Delta = \int_U e^{-tu_j} e^f dV_U \rightarrow \int_U e^{-tu} e^f dV_U (\leq \int_U e^{-(t+\epsilon)(\psi - \phi_L)} \mu_\Delta < \infty)$$

Since  $x$  was arbitrary, using a partition of unity, shows that  $I_t$  is continuous on  $PHS(X, L)$ . This proves the first continuity statement in the corollary, which in turn immediately imply the two points (for the second point one just uses the compactness of  $PSH(X, L) \cap \{\sup_X(\cdot - \phi_L)\}$ ).  $\square$

The next proposition allows one to write  $\alpha(L, \mu_\Delta)$  as a limit of log canonical thresholds of divisors. The proof is essentially the same as in the case when  $\Delta$  is trivial (or defines an orbifold divisor) recently considered by Demailly [36],

**Proposition 6.3.** *The functional  $I_t$  above is finite on  $PHS(X, L)$  iff it is finite on the subspace of all singular weights of the form  $\psi = \frac{1}{m} \log |s_m|^2$  for  $s_m \in H^0(X, mL)$ , where  $m$  is positive integer. In particular,  $\alpha(L, \mu_\Delta)$  is the sup over all such  $t$ .*

*Proof.* Take an arbitrary  $\psi \in PSH(X, L)$  and let

$$(6.1) \quad \|s_m\|_{(m\psi, \mu_\Delta)}^2 := \int_X |s_m|^2 e^{-m\psi} \mu_\Delta,$$

which defines a Hilbert norm on the  $N_m$ -dimensional subspace  $\mathcal{H}_m$  of  $H^0(X, mL)$  where it is finite. Next, let  $\psi_m \in PSH(X, L)$  be the weight of the Bergman metric defined by  $\mathcal{H}_m$ , i.e.

$$\psi_m := \frac{1}{m} \sup_{s_m \in H^0(X, mL)} \log \frac{|s_m|^2}{\|s_m\|_{(m\psi, \mu_\Delta)}^2} = \frac{1}{m} \log \sum_{i=1}^{N_m} |s_m^{(i)}|^2$$

where  $s_m^{(i)}$  is an orthonormal base for  $\mathcal{H}_m$ . Let  $\alpha_m := \sup \{t : I_t(\psi_m) < \infty\}$ . Then

$$(6.2) \quad 1/\alpha(L, \mu_\Delta) \leq 1/\alpha_m + 1/m$$

To see this we write  $e^{-\frac{m}{p}(\psi - \phi_L)} = e^{\frac{m\psi_m - m\psi}{p}} e^{\frac{-(m\psi_m - m\phi_L)}{p}}$  for a fixed  $p > 1$  and apply Hölder's inequality with dual exponents  $(p, q)$  (i.e  $1/q = 1 - 1/p$ ), giving

$$\int e^{-\frac{m}{p}(\psi - \phi_L)} \mu_\Delta \leq \left( \int e^{m\psi_m - m\phi} \mu_\Delta \right)^{1/p} \left( \int e^{-\frac{mq}{p}(\psi_m - \phi_L)} \right)^{1/q}$$

By the second equality in the definition of  $\psi_m$  above the first factor is a constant ( $= N_m^{1/p}$ ) and the second factor is finite as long as  $\frac{mq}{p} < \alpha_m$  which is equivalent to  $(\frac{m}{p})^{-1} < \frac{1}{\alpha_m} + \frac{1}{m}$ , i.e.  $p < m\alpha_m + 1$ . Since  $p > 1$  was arbitrary this proves 6.2.

Now take  $t$  such that  $I_t$  is finite for all weights of the form  $\frac{1}{m} \log |s_m|^2$ . By the second equality in the definition of  $\psi_m$  above combined with the concavity of log we deduce that  $I_t(\psi_m)$  is finite for any  $m$  and hence  $\alpha(\psi_m) \geq t$ . All in all this means that  $\alpha(L, \mu_\Delta) \geq t(1 + \epsilon_m)$ , where  $\epsilon_m \rightarrow 0$  and hence letting  $m \rightarrow \infty$  finishes the proof of the proposition.  $\square$

Formulated in terms of log canonical thresholds (see [36]) the previous proposition gives, in the case when  $\psi_\Delta$  is defined by a sub klt divisor,

$$\alpha(L, \mu_\Delta) = \inf_{m, D} \text{lct}_X(X, mD + \Delta),$$

where  $m$  is a positive integer and  $D$  is the zero divisor of some  $s_m \in H^0(X, mL)$ .

*Remark 6.4.* The proof of the first statement in the previous proposition only used that the measure  $\mu$  is such that the norm 6.1 is non-degenerate where it is finite, which for example holds for any  $\mu$  which does not charge pluripolar sets.

## REFERENCES

- [1] Adams D.R. A sharp inequality of J. Moser for higher order derivatives, *Ann. of Math.* 128 (1988), no. 2, 385–398.
- [2] Aubin, T.: Equations du type Monge-Ampère sur les variétés kählériennes compactes, *Bull. Sci. Math.* 102 :1 (1978) 63-95
- [3] Bando, S; Mabuchi, T: Uniqueness of Einstein Kahler metrics modulo connected group actions, *Algebraic geometry, Sendai, 1985*, *Adv. Stud. Pure Math.*, vol. 10, North-Holland, Amsterdam, 1987, pp. 11-40.
- [4] Bando, S; Kobayashi, R: Ricci-flat Kähler metrics on affine algebraic manifolds II. *Math. Ann.* 287 (1990), pp. 175–180
- [5] Bakry, D; Emery, M: Diffusions hypercontractives. *Séminaire de Probabilités XIX*, Springer Lecture Notes in Math. 1123 (1985), 175-206
- [6] Beckner, W: Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality. *Annals of Math.* 138 (1993), 213-242.
- [7] Berman, R.J. Analytic torsion, vortices and positive Ricci curvature. arXiv:1006.2988
- [8] Berman, R.J.: Kähler-Einstein metrics emerging from free fermions and statistical mechanics. arXiv:1009.2942
- [9] Berman, R; Boucksom, S; *Invent. Math.* 181 (2010), no. 2, 337-394
- [10] Berman, R.J.: Boucksom, S; Guedj, V; Zeriahi, A: A variational approach to complex Monge-Ampère equations. arXiv:0907.4490
- [11] Berman, R.J.; Boucksom, S; Witt Nyström, D: Fekete points and equidistribution on complex manifolds. *Acta Math.* (to appear). Preprint at arXiv:0907.2820
- [12] Berman, R.J.; Demailly, J-P: Regularity of plurisubharmonic upper envelopes in big cohomology classes. *Arkiv för Matematik* (to appear) arXiv:0905.1246
- [13] Berndtsson, B: A Brunn-Minkowski type inequality for Fano manifolds and the Bando-Mabuchi uniqueness theorem. arXiv:1103.0923
- [14] Boucksom, S; Essidieux, P; Guedj, V; Zeriahi: Monge-Ampère equations in big cohomology classes. *Acta. Math.* (to appear). arXiv:0812.3674
- [15] Blocki, Z: Uniqueness and stability for the Monge-Ampère equation on compact Kähler manifolds, *Indiana Univ. Math. J.* 52 (2003), no. 6 1697-1701.
- [16] Blocki, Z: The Calabi-Yau theorem, to appear in *Lecture Notes in Mathematics* as a part of the volume *Complex Monge-Ampère equations and geodesics in the space of Kähler metrics* (ed. V. Guedj). <http://gamma.im.uj.edu.pl/~blocki/publ/>
- [17] Blocki, Z; Kołodziej, S: On regularization of plurisubharmonic functions on manifolds, *Proceedings of the American Mathematical Society* 135 (2007), 2089-2093.
- [18] Boyer, C.P.; Galicki, K; Kollár, J: Einstein Metrics on Spheres: *Annals of Mathematics* 162(1) (2005), 557-580.
- [19] Brendle, S: Ricci flat Kahler metrics with edge singularities. arXiv:1103.5454
- [20] Calabi, E: Extremal Kähler metrics, in *Seminar on Differential Geometry*, volume 102 of *Ann. of Math. Stud.*, pages 259–290, Princeton Univ. Press, Princeton, N.J., 1982.
- [21] Calabi, E; Chen, X.X.: The space of Kähler metrics, 2, *J. Differential Geom.*, 61(2):173–193, 2002.
- [22] Campana, F; Guenancia, H; Păun, M: Metrics with cone singularities along normal crossing divisors and holomorphic tensor fields. arXiv:1104.4879
- [23] Cao, H.D: Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds. *Invent. Math.* 81 (1985), no. 2, 359–372.

- [24] Cao, Huai-Dong Existence of gradient Kähler-Ricci solitons. Elliptic and parabolic methods in geometry (Minneapolis, MN, 1994), 1–16,
- [25] Carlen, E and Loss, M: Competing symmetries, the logarithmic HLS inequality and Onofri's inequality on  $S_n$ . Geometric and Functional Analysis 2 (1992) 90–104.
- [26] Chen, X.X: On the lower bound of the Mabuchi energy and its application. Internat. Math. Res. Notices 2000, no. 12, 607–623.
- [27] Chen, X-X.: A New Parabolic Flow in Kähler Manifolds. Comm. Anal. Geom. 12 (2004), no. 4,
- [28] Chen, X-X. Space of Kähler metrics III– The greatest lower bound of the Calabi energy, Invent. math. 175, 453–680(2009).
- [29] Chen, X.X; He, W.Y: On the Calabi Flow. Amer. J. Math. Vol. 130, Num. 2, 1 2008
- [30] Chen, X. X.; Tian, G; Zhang, Z: On the weak Kähler-Ricci flow. arXiv:0802.0809.
- [31] Chrusciel, P.T.: Semi-global existence and convergence of solutions of the Robinson- Trautman (2-dimensional Calabi) equation, Comm. Math. Phys. 137 (1991), no. 2, 289–313.
- [32] Cianchi, A: Moser-Trudinger trace inequalities. Adv. in Math. Vol. 217, Issue 5, 20 2008, p. 2005–2044
- [33] Dembo, A; Zeitouni, O: Large deviations techniques and applications. Corrected reprint of the second (1998) edition. Stochastic Modelling and Applied Probability, 38. Springer-Verlag, Berlin, 2010.
- [34] Demailly, J. P.: Regularization of closed positive currents and intersection theory. J. Alg. Geom. 1 (1992), no. 3, 361–409.
- [35] Demailly, J-P; Kollar, J: Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds. Ann. Sci. École Norm. Sup. (4) 34 (2001), no. 4, 525–556.
- [36] Demailly, J-P: Appendix to I. Cheltsov and C. Shramov's article "Log canonical thresholds of smooth Fano threefolds" : On Tian's invariant and log canonical thresholds. Uspekhi Mat. Nauk, 63:5(383) (2008), 73–180
- [37] Demailly, J.-P.: Complex analytic and algebraic geometry; manuscript Institut Fourier, first edition 1991, available online at <http://www-fourier.ujf-grenoble.fr/~demailly/books.html> .
- [38] Dinh, T-CV; Nguyen, V-A; Sibony, N: Exponential estimates for plurisubharmonic functions. J. Differential Geom. Volume 84, Number 3 (2010), 465–488.
- [39] Ding, W.: Remarks on the existence problem of positive Kähler-Einstein metrics. Math. Ann. 463–472 (1988).
- [40] Ding, W. and Tian, G.: The generalized Moser-Trudinger Inequality. Proceedings of Nankai International Conference on Nonlinear Analysis, 1993.
- [41] Donaldson, S.K: Symmetric spaces, Kähler geometry and Hamiltonian dynamics. Northern California Symplectic Geometry Seminar, 13–33, Amer. Math. Soc. Transl. Ser. 2, 196, Amer. Math. Soc., Providence, RI, 1999.
- [42] Donaldson, S.K.: Moment maps and diffeomorphisms. Asian Journal of Math., vol. 3, pp. 1–16, 1999.
- [43] Donaldson, S.K.: Conjectures in Kähler geometry, in Strings and geometry, 71–78, Clay Math. Proc. 3, AMS 2004.
- [44] Donaldson, S.K.: "Discussion of the Kahler-Einstein problem". Available at <http://www2.imperial.ac.uk/~skdona/KENOTES.PDF>
- [45] Donaldson, S.K.: Kahler metrics with cone singularities along a divisor. arXiv:1102.1196
- [46] Eyssidieux, E; Guedj, E; Zeriahi, A: Singular Kähler-Einstein metrics. arXiv:math/0603431. J. Amer. Math. Soc. 22 (2009), 607–639.
- [47] Fine, J., Constant scalar curvature Kähler metrics on fibred complex surfaces, J. Differential Geom. 68 (2004), no. 3, 397–432.
- [48] Fontana, L.: Sharp borderline Sobolev inequalities on compact Riemannian manifolds, Comment. Math. Helv. 68 (1993), 415–454.
- [49] Fontana, L; Morpurgo, C: Adams inequalities on measure spaces. arXiv:0906.5103
- [50] Ghigi, A; Kollar, J: Kaehler-Einstein metrics on orbifolds and Einstein metrics on spheres. Comment. Math. Helv., 82(4), 877–902 (2007)
- [51] Guedj,V; Zeriahi, A: Intrinsic capacities on compact Kähler manifolds. J. Geom. Anal. 15 (2005), no. 4, 607–639.
- [52] Guedj,V; Zeriahi, A: The weighed Monge-Ampère energy of quasiplurisubharmonic functions. J. Funct. An. 250 (2007), 442–482.
- [53] Hiep, P.H.: Holder continuity of solutions to the complex Monge-Ampere equations on compact Kahler manifolds. . arXiv:0904.4145. To appear in Annales de l'Institut Fourier

- [54] Jeffres, T: Uniqueness of Kähler-Einstein cone metrics, *Publ. Mat.* 44, 437–448 (2000)
- [55] Jeffres, T: Schwarz Lemma for Kähler Cone Metrics. *Int Math Res Notices* (2000).
- [56] Jeffres, T; Mazzeo, R; Rubinstein, Y.A.; Kähler-Einstein metrics with edge singularities. *Arxiv* 1105.5216
- [57] Kazdan; Warner, F. W: Curvature functions for open 2-manifolds, *Ann. of Math.* 99 (1974) 203-219.
- [58] Kołodziej, S.: The complex Monge–Ampère equation. *Acta Math.* 180, 69–117 (1998).
- [59] Kołodziej, S.: Hölder continuity of solutions to the complex Monge–Ampère equation with the right-hand side in  $L^p$ : the case of compact Kähler manifolds
- [60] C. Li: On the limit behaviour of metrics in continuity method to Kahler-Einstein problem in toric Fano case. *arxiv* 1012.5229
- [61] C. Li: Remarks on logarithmic K-stability. *arXiv:1104.0428*
- [62] Li, H: On the lower bound of the K-energy and F-functional. *Osaka J. Math.* Volume 45, Number 1 (2008), 253-264.
- [63] Luo; F; Tian, G: Liouville equation and spherical convex polytopes. *Proc. Amer. Math. Soc.* 116 (1992), 1119-1129.
- [64] Mabuchi, T: K-energy maps integrating Futaki invariants. *Tohoku Math. J. (2)* 38 (1986), no. 4, 575–593.
- [65] Mabuchi, T: Some symplectic geometry on compact Kähler manifolds. I, *Osaka Journal of Mathematics* 24 (1987), 227–252.
- [66] Mazzeo, R: Elliptic theory of differential edge operators, I, *Comm. PDE* 16 (1991), 1616-1664
- [67] Mazzeo, R: Kähler-Einstein metrics singular along a smooth divisor. *Journées "Équations aux Dérivées Partielles"* (Saint-Jean-de-Monts, 1999), Exp. No. VI, 10 pp., Univ. Nantes, Nantes, 1999.
- [68] Nadel, A.M.: Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature, *Annals of Mathematics* 132 (1990), 549-596.
- [69] Paun, M: Relative critical exponents, non-vanishing and metrics with minimal singularities. *Inventiones Mathematicae* (2011), pp. 1-64
- [70] Phong, D.H: Song, J; Sturm, J; Weinkove, B: The Moser-Trudinger inequality on Kahler-Einstein manifolds. *Amer. J. Math.* 130 (2008), no. 4, 1067-1085, *arXiv:math/0604076*
- [71] Rubinstein, Y.A: On energy functionals, Kahler-Einstein metrics, and the Moser-Trudinger-Onofri neighborhood, *J. Funct. Anal.* 255, special issue dedicated to Paul Malliavin (2008), 2641-2660.
- [72] Rubinstein, Y.A: On the construction of Nadel multiplier ideal sheaves and the limiting behavior of the Ricci flow. *Trans. Amer. Math. Soc.* 361 (2009), 5839-5850.
- [73] Rubinstein, Y.A: Some discretizations of geometric evolution equations and the Ricci iteration on the space of Kahler metrics *Adv. Math.* 218 (2008), 1526-1565.
- [74] Song, J; Tian, G: The Kähler-Ricci flow on surfaces of positive Kodaira dimension. *Inv. Math.* 170 (2007), 609-653.
- [75] Song, J; Tian, G: Canonical measures and Kahler-Ricci flow. *arXiv:0802.2570*
- [76] Song, J; Tian, G: The Kahler-Ricci flow through singularities. *arXiv:0909.4898*
- [77] Stoppa, J: Twisted constant scalar curvature Kähler metrics and Kähler slope stability. *J. Differential Geom.* Volume 83, Number 3 (2009), 663-691.
- [78] Székelyhidi, G: The Calabi functional on a ruled surface. *Ann. Sci. Éc. Norm. Supér.* 42 (2009), 837–856
- [79] Székelyhidi, G; Tosatti.V: Regularity of weak solutions of a complex Monge-Ampère equation. To appear in *Analysis & PDE*
- [80] Ross, J; Thomas, R. P.: Weighted projective embeddings, stability of orbifolds and constant scalar curvature Kähler metrics. *arXiv:0907.5214*
- [81] Tarantello, G: Selfdual gauge field vortices. An analytical approach. *Progress in Nonlinear Differential Equations and their Applications*, 72. Birkhäuser Boston, Inc., Boston, MA, 2008. xiv+325 pp.
- [82] Tosatti.V; Weinkove, B: The Calabi flow with small initial data. *Math. Res. Lett.* 14 (2007), no. 6, 1033–1039
- [83] Tian, G: On Kähler-Einstein metrics on certain Kähler manifolds with  $C_1(M) > 0$ , *Inventiones Mathematicae* 89 (1987), 225–246.
- [84] Tian, G; On Calabi's conjecture for complex surfaces with positive first Chern class. *Invent. Math.* Vol. 101, Nr. 1 (1990)



- [85] Tian, G. Kähler-Einstein metrics with positive scalar curvature, *Invent. Math.* 130 (1997), no. 1, 1–37.
- [86] Tian, G: *Canonical Metrics in Kähler Geometry*, Birkhäuser, 2000.
- [87] Tian, G: Kähler-Einstein metrics on algebraic manifolds, in: *Transcendental methods in algebraic geometry* (Cetraro 1994), 143–185.
- [88] Tian, G; Zhu, X: Convergence of Kähler-Ricci flow. *J. Amer. Math. Soc.* 20 (2007), no. 3
- [89] Tian, G; Yau, S-T: Complete Kähler manifolds with zero Ricci curvature, I. *J. Amer. Math. Soc.* 3 (1990), no. 3, 579–609.
- [90] Troyanov, M: Metrics of constant curvature on a sphere with two conical singularities. *Differential geometry, Proc. 3rd International Symposium on Differential Geom. (Peniscola, 1988)*, *Lecture Notes in Math*, Vol 1410, Springer-Verlag, 296-308
- [91] Troyanov, M: Prescribing curvature on compact surfaces with conical singularities, *Trans. AMS*, 324 (1991) 793-821
- [92] Tsuji, H: Ricci iterations and canonical Kähler-Einstein currents on log canonical pairs. *arXiv:0903.5445*
- [93] Yau, S.T: On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. *Comm. Pure Appl. Math.* 31 (1978), no. 3, 339–41
- [94] Zeriahi, A: Volume and capacity of sublevel sets of a Lelong class of psh functions. *Indiana Univ. Math. J.* 50 (2001), no. 1, 671–703.

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